# Orthosymplectic, Periplectic, and Twisted Super Yangians

by

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# Abstract

This dissertation establishes various structural and representation theoretic results in super Yangian theory.

In its first part, this dissertation details the algebraic structure and representation theory for the Yangians of orthosymplectic Lie superalgebras. Addressing these Yangians via the $RTT$ realization, we prove a Poincaré-Birkhoff-Witt-type theorem and provide a thorough study of the algebraic structure of their extended Yangians. The main result of this part, and of this dissertation, is the provision of many necessary conditions for the irreducible representations of these orthosymplectic Yangians to be finite-dimensional; furthermore, there is much progress made to address attaining sufficient conditions as well. These representation theoretic results are accomplished via the development of a highest weight theory, and such necessary conditions are given in terms of highest weights and tuples of Drinfel'd polynomials.

The second part of this dissertation is devoted to the Yangians of periplectic Lie superalgebras and the twisted Yangians associated to symmetric superpairs of type AIII.

Via the $RTT$ formalism, we prove many structural results for the Yangians of type $P$ strange Lie superalgebras that have only so far been established for the Yangians of type $Q$ strange Lie superalgebras, including a proof of a Poincaré-Birkhoff-Witt-type theorem.

The twisted super Yangians of type AIII are defined along with many structural properties established. We lay the foundation for the classification of their finite-dimensional irreducible representations by cultivating a highest weight theory and proving that all finite-dimensional irreducible modules must be highest weight.

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# Contents

# List of Figures

# List of Symbols

# Chapter 1

# Introduction

Yangians comprise one of the two important families of affine quantum groups, alongside the Drinfel'd-Jimbo quantum affine algebras. As formalized by Vladimir Drinfel'd in his seminal paper [Dri85], the Yangian $\mathrm{Y}_\hbar(\mathfrak{g})$ of a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ is a certain non-commutative and non-cocommutative Hopf algebra over $\mathbb{C}[\hbar]$ that quantizes a canonical Lie bialgebra structure on the polynomial current Lie algebra $\mathfrak{g}[z]$, which, among other requirements, means:

$$\mathrm{Y}_\hbar(\mathfrak{g})/\hbar\,\mathrm{Y}_\hbar(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{g}[z]).$$

One often aims to study the Yangian when the parameter $\hbar$ takes on a nonzero complex number $\lambda \in \mathbb{C}^*$, therefore working over the field $\mathbb{C}$ in lieu of $\mathbb{C}[\hbar]$. In this case, the Yangian is commonly denoted $\mathrm{Y}(\mathfrak{g})$ since $\mathrm{Y}_\lambda(\mathfrak{g})$ is isomorphic to $\mathrm{Y}_1(\mathfrak{g})$ for all such values $\lambda$. In particular, it is a foundational theorem of Drinfel'd in his aforementioned paper that every finite-dimensional irreducible representation of the Yangian $\mathrm{Y}(\mathfrak{g})$ yields a rational solution to the quantum Yang-Baxter equation (QYBE):

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

As an essential consistency equation for integrable models in statistical mechanics and quantum field theory (see [KS82b, Jim89], for instance), the search for non-trivial rational solutions to the QYBE serves as the primary motivation for investigating Yangians and exploring their representation theory. However, it is known that Yangians originally arose prior to their formalization by Drinfel'd; namely, the Yangian of $\mathfrak{gl}_N$

emerged in work on the quantum inverse scattering method, which describes a systematic approach for solving certain integrable systems (refer to [TF79, KS82a]).

The structure and representation theory of Yangians, as well as their extensions, based on finite-dimensional complex simple Lie algebras has been extensively studied in Drinfel'd's papers [Dri85, Dri86b, Dri88] and the following: [CP95, MNO96, Mol07, AMR06, Wen18, GNW18, GRW19a, GLW21]. In fact, the complete classification of their finite-dimensional irreducible representations was established by Drinfel'd in [Dri88], with a more detailed exposition available in [AMR06, Mol07].

Since their formulation, the theory of Yangians has expanded to include constructions based on the general linear Lie superalgebras and the (non-exceptional) finite-dimensional classical Lie superalgebras as described by Victor Kac [Kac77]. Hence, the natural questions raised in this context are whether one can extend Drinfel'd's fundamental theorems for the Yangian to the supersymmetric setting and whether it is possible to achieve a classification of the finite-dimensional irreducible representations for such *super* Yangians. In fact, for Yangians based on the general linear Lie superalgebras, a positive answer to the latter question was provided in the 1990's by two articles of Ruibin Zhang (see [Zha95, Zha96]).

The core property of the Yangian that is pivotal to Drinfel'd's theorems is the existence of *universal R-matrix*

$$\mathcal{R}(u) = \mathbf{1} + \sum_{k=1}^{\infty} \mathcal{R}_k u^{-k} \in (\mathrm{Y}(\mathfrak{g}) \otimes \mathrm{Y}(\mathfrak{g}))[\![u^{-1}]\!].$$

The question of whether such a universal $R$-matrix exists for super Yangians remains open; however, for reasons explained in § 1.1, there may be good reason to believe that progress on this problem is achievable at least when $\mathfrak{g}$ is an orthosymplectic Lie superalgebra.

In this dissertation, we address three different topics in super Yangian theory, demarcated in two parts. The first part, comprising Chapters 2 and 3, focuses on Yangians of orthosymplectic Lie superalgebras, while the second part, encompassing Chapters 4 and 5, considers Yangians of the strange Lie superalgebras and twisted super Yangians of type AIII, respectively.

In particular, Chapter 2 is dedicated to establishing many structural results for the orthosymplectic Yangians, as originally defined in [AAC+03]. The primary results of this chapter consists of a detailed proof of a Poincaré-Birkhoff-Witt-type theorem for these Yangians and a thorough study of their extended Yangians. Chapter 3 investigates the representation theory of these Yangians and contains our approach to addressing a classification of their finite-dimensional irreducible representations. A summary of these results, and how they are achieved, is provided in §1.1 below.

Yangians of the strange Lie superalgebras were first defined by Maxim Nazarov in [Naz92], with a more detailed exposition on the type $Q$ case published later in [Naz99]. The purpose of Chapter 4 is to adapt and prove many of the structural results of Nazarov's latter paper in the type $P$ case, including a proof of a Poincaré-Birkhoff-Witt-type theorem. We refer the reader to §1.2 for our main results on this topic. Finally, Chapter 5 introduces the notion of twisted super Yangians as a direct super-analogue of those twisted Yangians defined in [MR02]. Among various structural results, we lay the foundation for the classification of their finite-dimensional irreducible representations by establishing a highest weight theory and proving that all finite-dimensional irreducible modules must be highest weight: see §1.3 for a survey of these results.

## 1.1 Yangians of Orthosymplectic Lie Superalgebras

### 1.1.1 Background and motivation

Yangians of orthosymplectic Lie superalgebras were first defined in [AAC+03] utilizing the $RTT$ formalism, which we will deploy in this dissertation. However, we should mention that traditionally, Yangians of finite-dimensional simple Lie algebras admit at least three important presentations: Drinfel'd's $J$-presentation, Drinfel'd's current presentation, and the $RTT$ realization (see [Dri85, Dri88, RTF16], respectively).

Vladimir Drinfel'd's consequential theorems regarding Yangians of finite-dimensional simple Lie algebras were originally proven in terms of his original $J$-presentation, including the existence of the universal $R$-matrix. Many of his results, including the construction of the universal $R$-matrix, were later published in more detail in [GLW21], but in terms of Drinfel'd's current presentation. It is worth noting that only recently has progress been made in establishing Drinfel'd's current presentation for the Yangians of orthosymplectic Lie superalgebras (refer to [Mol23a, MR23]). This development has allowed for the possibility to address the construction of the universal $R$-matrix in the supersymmetric case; however, undertaking this particular question will be outside the scope of this dissertation.

Returning to the *RTT* realization, it is natural to define the Yangian as a certain quotient of its *extended* Yangian. Namely, suppose $\mathbb{C}^{M|N}$ denotes the super vector space $\mathbb{C}^{M+N}$ whose first $M$ standard basis vectors are even while the rest are odd. Allowing $R(u) \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}(u)$ to denote the solution to the (super) QYBE given by (2.2.3), one may define the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ of the orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|N}$ as the Hopf superalgebra whose generators are collected into a matrix

$$T(u) \in \operatorname{End}(\mathbb{C}^{M|N}) \otimes \mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$$

subject to defining relations as described by the *RTT-relation*:

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v).$$

As shown in [AAC+03], there exists a formal series $\mathcal{Z}(u) = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Z}_n u^{-n}$ consisting of even central elements in the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$, allowing for one to define the quotient

$$\mathrm{Y}(\mathfrak{osp}_{M|N}) := \mathrm{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u) - \mathbf{1}),$$

where $(\mathcal{Z}(u) - \mathbf{1})$ denotes the graded ideal generated by the set $\{\mathcal{Z}_n \mid n \in \mathbb{Z}^+\}$. In particular, it can be shown that $(\mathcal{Z}(u) - \mathbf{1})$ is a graded Hopf ideal, therefore endowing the Yangian with an induced Hopf superstructure as well. The above definition proves to be appropriate due to the fact that its Rees superalgebra $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) := R_\hbar(\mathrm{Y}(\mathfrak{osp}_{M|N}))$ serves as a homogeneous quantization of a canonical Lie superbialgebra structure on the polynomial current Lie superalgebra $\mathfrak{osp}_{M|N}[z]$ (refer to §2.3.3 for a detailed exposition on this point).

There are, however, several questions concerning the algebraic structure of both the Yangian and the extended Yangian of $\mathfrak{osp}_{M|N}$ that are yet to be proven in detail in the literature:

(i) Describe explicit algebraic bases for $\mathrm{Y}(\mathfrak{osp}_{M|N})$ and $\mathrm{X}(\mathfrak{osp}_{M|N})$.

(ii) Determine the (super)centers of $\mathrm{Y}(\mathfrak{osp}_{M|N})$ and $\mathrm{X}(\mathfrak{osp}_{M|N})$.

(iii) Prove $\mathrm{Y}(\mathfrak{osp}_{M|N})$ is isomorphic to the subalgebra of $\mathrm{X}(\mathfrak{osp}_{M|N})$ fixed by a family of automorphisms $\{\mu_f\}_f \subset \operatorname{Aut}(\mathrm{X}(\mathfrak{osp}_{M|N}))$ indexed by $f = f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$:

$$\mathrm{Y}(\mathfrak{osp}_{M|N}) \cong \left\{Y \in \mathrm{X}(\mathfrak{osp}_{M|N}) \mid \mu_f(Y) = Y \;\text{ for all }\; f = f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]\right\}.$$

Addressing such structural questions is the main purpose of Chapter 2. Of course, the primary motivation for studying these Yangians is the investigation of their representation theory, and in particular, the determination of the isomorphism classes of their finite-dimensional irreducible representations, denoted

$$\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{Y}(\mathfrak{osp}_{M|N}))_{/\sim} \quad \text{and} \quad \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N}))_{/\sim}.$$

These investigations comprise Chapter 3; however, we note that classifications of the above sets has recently been accomplished in the cases $M = 1$ and $M = 2$ by virtue of Alexander Molev's recent papers (see [Mol21, Mol23b, Mol22b]). In due course, we will present his theorems after introducing the necessary notation in the subsequent subsection.

### 1.1.2 Main results

We shall now summarize the main findings from Chapters 2 and 3, starting with the structural results that comprise Chapter 2. The primary realization of this first chapter on orthosymplectic Yangians is the statement of a Poincaré-Birkhoff-Witt-type theorem as described in §2.3. We present the PBW-type theorem given as part (a) of Theorem I which asserts that the associated graded superalgebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{osp}_{M|N})$ of the Yangian, with respect to a certain filtration, must have a basis by the PBW Theorem for Lie superalgebras; hence, a basis is provided for the Yangian as well. We also present parts (b) and (c), which are immediate consequences of the first part of the theorem:

**Theorem I.** *The Yangian* $\mathrm{Y}(\mathfrak{osp}_{M|N})$ *has the following structural properties:*

(a) *There is an* $\mathbb{N}$*-graded Hopf superalgebra isomorphism*

$$\mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \cong \mathrm{gr}\,\mathrm{Y}(\mathfrak{osp}_{M|N}).$$

(b) *The supercenter* $\mathrm{ZY}(\mathfrak{osp}_{M|N})$ *of* $\mathrm{Y}(\mathfrak{osp}_{M|N})$ *is trivial:* $\mathbb{C}\cdot\mathbf{1}$*.*

(c) *There is a Hopf superalgebra embedding*

$$\iota\colon \mathfrak{U}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathrm{Y}(\mathfrak{osp}_{M|N}).$$

The conglomerated results in Theorem I can all be found in § 2.3 as stated in

Theorem 2.3.3, Corollary 2.3.5, and Proposition 2.3.6, respectively.

The second set of structural results are for the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$. Recalling there exists a formal series $\mathcal{Z}(u) = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Z}_n u^{-n}$ consisting of even central elements in $\mathrm{X}(\mathfrak{osp}_{M|N})$, we have the following:

**Theorem II.** *The extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ has the following structural properties:*

(a) *There is a Hopf superalgebra isomorphism*

$$\mathrm{X}(\mathfrak{osp}_{M|N}) \cong \mathbb{C}[\mathcal{Z}_n \mid n \in \mathbb{Z}^+] \otimes \mathrm{Y}(\mathfrak{osp}_{M|N}).$$

(b) *The supercenter $\mathrm{ZX}(\mathfrak{osp}_{M|N})$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$ is $\mathbb{C}[\mathcal{Z}_n \mid n \in \mathbb{Z}^+]$.*

(c) *There is an $\mathbb{N}$-graded Hopf superalgebra isomorphism*

$$\mathfrak{U}(\mathfrak{osp}_{M|N}[z] \oplus \mathbb{C}[z]) \cong \mathrm{gr}\, \mathrm{X}(\mathfrak{osp}_{M|N}).$$

(d) *There is a Hopf superalgebra embedding*

$$\iota \colon \mathfrak{U}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathrm{X}(\mathfrak{osp}_{M|N}).$$

(e) *There is a family of automorphisms $\{\mu_f\}_f \subset \mathrm{Aut}(\mathrm{X}(\mathfrak{osp}_{M|N}))$ indexed by the collection $f = f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that:*

$$\mathrm{Y}(\mathfrak{osp}_{M|N}) \cong \left\{ Y \in \mathrm{X}(\mathfrak{osp}_{M|N}) \mid \mu_f(Y) = Y \;\; \textit{for all} \;\; f = f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!] \right\}.$$

All the results in Theorem II are described in §2.4. Respectively, these are given by Theorem 2.4.2, Proposition 2.4.3, Theorem 2.4.4, Proposition 2.4.6, and Theorem 2.4.7. We note that Part (c) in Theorem II provides a Poincaré-Birkhoff-Witt-type theorem for the extended Yangian.

Chapter 3 starts by investigating a highest weight theory for $\mathrm{X}(\mathfrak{osp}_{M|N})$ based on fixing a certain positive root system $\Phi^+$ for the orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|N}$. The definitions that form the foundation of the theory are found in §3.1.2, which in summary describe that highest weights take the form of tuples of power series

$$\lambda(u) = (\lambda_k(u))_{k=1}^{M+N} \in \prod_{k=1}^{M+N} \left(1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]\right).$$

To any such highest weight $\lambda(u)$, one may define a Verma module $M(\lambda(u))$ and, provided it is non-trivial, it will have an irreducible quotient $L(\lambda(u))$.

In particular, we will see that all such finite-dimensional irreducible quotients $L(\lambda(u))$ will exhaust the set $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N}))_{/\sim}$ via their isomorphism classes, so the characterization of the non-triviality of $M(\lambda(u))$ is one of the first important tasks of Chapter 3. In fact, A. Molev in [Mol23b, Mol22b] has proven such characterization in the cases $M = 1$ and $M = 2$, as determined by the following theorem:

**Theorem M1.** *Set $M = 1$ or $M = 2$, $N \geq 2$, and assign $m = \lfloor \frac{M}{2} \rfloor$, $n = \frac{N}{2}$, $\kappa = (M-N-2)/2$. The Verma module $M(\lambda(u))$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$ is non-trivial if and only if its highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfies the consistency conditions*

$$\frac{\lambda_{M+j}(u)}{\lambda_{M+j+1}(u)} = \frac{\lambda_{M+N-j}(u-\kappa-j+m)}{\lambda_{M+N+1-j}(u-\kappa-j+m)} \quad \textit{for} \quad j = 1, 2, \ldots, n-1,$$

*and*

$$\frac{\lambda_1(u)}{\lambda_{n+1}(u)} = \frac{\lambda_{n+2}(u-\kappa-n)}{\lambda_1(u-\kappa-n)} \quad \textit{when} \quad M = 1,$$

*or*

$$\frac{\lambda_1(u)}{\lambda_3(u)} = \frac{\lambda_{2+N}(u-\kappa+1)}{\lambda_2(u-\kappa+1)} \quad \textit{when} \quad M = 2.$$

*Moreover, for every finite-dimensional irreducible representation $V$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$, it holds that*

$$V \cong L(\lambda(u))$$

*for some unique tuple $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfying the above relations. The highest weight vector of $V$ is unique up to scalar multiple.*

Section §3.1.3 is devoted to a pivotal construction in the representation theory of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$: the construction of non-trivial covariant functors

$$\begin{aligned} \mathcal{F}^+_M\colon \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N})) &\to \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{(M-2)|N})) \quad \text{for} \quad M \geq 2 \\ \text{and} \quad \mathcal{F}^N_+\colon \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N})) &\to \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|(N-2)})) \quad \text{for} \quad N \geq 2, \end{aligned}$$

which we refer to as *restriction functors*. In particular, via an inductive argument on the rank of $\mathfrak{osp}_{M|N}$ that utilizes the above restriction functors and takes Theorem M1 as the induction base, we are able to classify the necessary and sufficient conditions for the non-triviality of the Verma module $M(\lambda(u))$ in full generality:

**Theorem III.** *Set $N \geq 2$ and let $m = \lfloor \frac{M}{2} \rfloor$, $\widehat{m} = \lceil \frac{M}{2} \rceil$, $n = \frac{N}{2}$, $\kappa = (M-N-2)/2$. The Verma module $M(\lambda(u))$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$ is non-trivial if and only if its highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfies the consistency conditions*

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{\lambda_{M-i}(u-\kappa+i)}{\lambda_{M+1-i}(u-\kappa+i)} \quad \textit{for} \quad i = 1, 2, \ldots, m-1,$$

$$\frac{\lambda_{M+j}(u)}{\lambda_{M+j+1}(u)} = \frac{\lambda_{M+N-j}(u-\kappa-j+m)}{\lambda_{M+N+1-j}(u-\kappa-j+m)} \quad \textit{for} \quad j = 1, 2, \ldots, n-1,$$

*and when $M$ is odd:*

$$\frac{\lambda_m(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}(u-\kappa+m)}{\lambda_{\widehat{m}+1}(u-\kappa+m)} \quad \textit{if } M \geq 3,$$

$$\textit{and} \quad \frac{\lambda_{\widehat{m}}(u)}{\lambda_{M+n}(u)} = \frac{\lambda_{M+n+1}(u-\kappa+m-n)}{\lambda_{\widehat{m}}(u-\kappa+m-n)},$$

*or when $M$ is even:*

$$\frac{\lambda_m(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}(u-\kappa+m)}{\lambda_{m+1}(u-\kappa+m)} \quad \textit{if } M \geq 2.$$

*Moreover, for every finite-dimensional irreducible representation $V$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$, it holds that*

$$V \cong L(\lambda(u))$$

*for some unique tuple $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfying the above relations. The highest weight vector of $V$ is unique up to scalar multiple.*

Theorem III is given as Theorem 3.2.2, where itself is a consequence of Theorem 3.1.3, Proposition 3.1.14, and Proposition 3.2.1. The next important task of Chapter 3 is to classify the conditions for when $L(\lambda(u))$ is finite-dimensional. As in A. Molev's aforementioned papers, this classification has been successfully completed in the cases $M = 1$ and $M = 2$:

**Theorem M2.** *Set $M = 1$ or $M = 2$, $N \geq 2$, $n = \frac{N}{2}$, and let $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfy the consistency conditions in Theorem M1. The $\mathrm{X}(\mathfrak{osp}_{M|N})$-module $L(\lambda(u))$ is finite-dimensional if and only if there exists a tuple of monic polynomials*

$$(\delta_{M2}\widetilde{Q}(u), \delta_{M2}Q(u); (P_k(u))_{k\in I}) \in \mathbb{C}[u]^{n+2\delta_{M2}},$$

*with $I = \{M+1, \ldots, M+n\}$, such that*

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u-1)}{P_k(u)} \quad \textit{for all} \quad k \in I \setminus \{M+n\},$$

*and*

$$\frac{\lambda_1(u)}{\lambda_{n+1}(u)} = \frac{P_{1+n}(u+1)}{P_{1+n}(u)} \quad \textit{when} \quad M = 1,$$

*or*

$$\frac{\lambda_{n+2}(u)}{\lambda_{n+3}(u)} = \frac{P_{2+n}(u-2)}{P_{2+n}(u)} \qquad \textit{and} \qquad \frac{\lambda_1(u)}{\lambda_3(u)} = \frac{\widetilde{Q}(u)}{Q(u)} \quad \textit{when} \quad M = 2,$$

*where $\widetilde{Q}(u)$ and $Q(u)$ are coprime polynomials of the same polynomial degree.*

Setting $(-1)^{[k]} = 1$ for $1 \leq k \leq M$ and $(-1)^{[k]} = -1$ for $M+1 \leq k \leq M+N$, we now state one of the main results of Chapter 3:

**Theorem IV.** *Set $M, N \geq 2$, $m = \lfloor \frac{M}{2} \rfloor$, $\widehat{m} = \lceil \frac{M}{2} \rceil$, $n = \frac{N}{2}$, and let $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfy the consistency conditions in Theorem III. If the $\mathrm{X}(\mathfrak{osp}_{M|N})$-module $L(\lambda(u))$ is finite-dimensional, then there exists a tuple of monic polynomials*

$$(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I}) \in \mathbb{C}[u]^{m+n+1},$$

*with $I = \{1, \ldots, m-1; M+1, \ldots, M+n\}$, such that*

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u+(-1)^{[k]})}{P_k(u)} \qquad \textit{for} \quad k \in I \setminus \{M+n\},$$

$$\frac{\lambda_{\widehat{m}}(u)}{\lambda_{M+n}(u)} = \frac{P_{M+n}(u+1)}{P_{M+n}(u)} \quad \textit{if} \quad M \textit{ is odd},$$

$$\frac{\lambda_{M+n}(u)}{\lambda_{M+n+1}(u)} = \frac{P_{M+n}(u-2)}{P_{M+n}(u)} \quad \textit{if} \quad M \textit{ is even},$$

*and*

$$\frac{\lambda_m(u)}{\lambda_{M+1}(u)} = \frac{\widetilde{Q}(u)}{Q(u)},$$

*where $\widetilde{Q}(u)$ and $Q(u)$ are coprime polynomials of the same polynomial degree.*

The polynomials $(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$ are called the *Drinfel'd polynomials* corresponding to $L(\lambda(u))$ and they are uniquely determined by the highest weight $\lambda(u)$. Given as Theorem 3.2.8, the proof of Theorem IV follows from the combination of 1) R. B. Zhang's classification of $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{Y}(\mathfrak{gl}_{m|n}))_{/\sim}$ (see [Zha96]) due to an embedding $\mathrm{Y}(\mathfrak{gl}_{m|n}) \hookrightarrow \mathrm{X}(\mathfrak{osp}_{M|N})$, and 2) an inductive argument on the rank of $\mathfrak{osp}_{M|N}$ via the use of restriction functors and taking Theorem M2 as the induction base.

Proving the converse of Theorem IV will be considerably more technical, which will involve the construction of *fundamental representations* of $\mathrm{X}(\mathfrak{osp}_{M|N})$ corresponding to Drinfel'd polynomials of the form

$$\big(u+\alpha, u+\beta; (1)_{k\in I}\big) \quad \text{or} \quad \big(1, 1; ((u+\gamma)^{\delta_{ik}})_{k\in I}\big)$$

for $i \in I$ and $\alpha, \beta, \gamma \in \mathbb{C}$ where $\alpha \neq \beta$. Those fundamental representations associated to the first tuple will be denoted $L(\lambda(u); \alpha, \beta)$, whereas those corresponding to the second tuple will be denoted $L(\lambda(u); i\colon \gamma)$. We now consider the following theorem which partially addresses the sufficiency of the conditions stated in Theorem IV.

**Theorem V.** *Set $M \geq 2$, let $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha \neq \beta$, and let $\lambda(u)$ satisfy the consistency conditions in Theorem III.*

(a) *Set $N \geq 2$. When $M = 2$, then $\dim L(\lambda(u); \alpha, \beta) \leq 2^N$. Otherwise when $M \geq 3$, then $\dim L(\lambda(u); \alpha, \beta) < \infty$ if and only if $\alpha - \beta \in O$, where $O$ is a certain non-trivial subset of $\frac{1}{2}\mathbb{Z}^+$. When $M$ is even in this case, then $\dim L(\lambda(u); \alpha, \beta) \leq 2^{mN}$.*

(b) *When $1 \leq i \leq m-1$, then $\dim L(\lambda(u); i\colon \gamma) < \infty$.*

We refer the reader to the subsections §3.2.3 and §3.2.4 for the results described by Theorem V. The notable omission in the above theorem is that there is no determination on the finite-dimensionality of $L(\lambda(u); i\colon \gamma)$ when $M+1 \leq i \leq M+n$. Moreover, further necessary conditions on the roots of the Drinfel'd polynomials $\widetilde{Q}(u)$ and $Q(u)$ that would be inferred by Theorem V have not yet been established. Currently, these questions remain open.

Refer to Conjectures 3.2.22 and 3.2.23 for our estimation on the classifications of the sets $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N}))_{/\sim}$ and $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{Y}(\mathfrak{osp}_{M|N}))_{/\sim}$, respectively.

## 1.2 Yangians of Strange Lie Superalgebras

### 1.2.1 Motivation and main results

In Kac's classification of finite-dimensional simple Lie superalgebras [Kac77], there are two families of finite-dimensional classical Lie superalgebras which are not basic: the simple strange Lie superalgebras of types $P$ and $Q$. Types $P$ and $Q$ each describe several families of finite-dimensional "strange" Lie superalgebras, including the aforementioned simple strange Lie superalgebras. In this work, we consider the most general families in each type, which we denote $\mathfrak{p}_N$ and $\mathfrak{q}_N$ for $N \in \mathbb{Z}^+$, respectively. We note that in regards to $\mathfrak{p}_N$ or $\mathfrak{q}_N$, the simple strange Lie superalgebras are either Lie sub-superalgebras or quotients of such.

In [Naz92], Nazarov defined Yangians for these strange Lie superalgebras $\mathfrak{p}_N$ and $\mathfrak{q}_N$, which are regarded as the Yangians of types $P$ and $Q$, respectively. Nazarov later studied the Yangians of type $Q$ more extensively in [Naz99], but there has been no comparable article for the Yangians of type $P$ released to date, prompting the motivation for this work.

In contrast to the finite-dimensional simple Lie algebras and the orthosymplectic Lie superalgebras, there is no canonical Lie superbialgebra structure on the polynomial current Lie superalgebra $\mathfrak{s}_N[z]$ for $\mathfrak{s}_N = \mathfrak{p}_N, \mathfrak{q}_N$. As a consequence of this, Nazarov proposed the following construction: by regarding $\mathfrak{s}_N$ as a fixed-point Lie sub-superalgebra $\mathfrak{gl}_{N|N}^{\vartheta}$ of $\mathfrak{gl}_{N|N}$ under a certain involution $\vartheta \in \mathrm{Aut}(\mathfrak{gl}_{N|N})$, one can extend $\vartheta$ in a non-trivial way to an involutive automorphism of $\mathfrak{gl}_{N|N}[z]$ via the assignment

$$\vartheta(f(z)) = \vartheta(f)(-z) \quad \text{for all} \quad f(z) \in \mathfrak{gl}_{N|N}[z].$$

In particular, one is able to define a natural Lie superbialgebra structure on the *twisted* polynomial current Lie superalgebra $\mathfrak{gl}_{N|N}[z]^{\vartheta}$, the fixed-point Lie sub-superalgebra of $\mathfrak{gl}_{N|N}[z]$ under the involution $\vartheta$ (refer to §4.3.3 for a detailed discussion on this point); namely, the Yangian $\mathrm{Y}_\hbar(\mathfrak{s}_N)$ will (homogeneously) quantize this Lie superbialgebra structure:

$$\mathrm{Y}_\hbar(\mathfrak{s}_N)/\hbar\,\mathrm{Y}_\hbar(\mathfrak{s}_N) \cong \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\vartheta}).$$

In Nazarov's second paper, he proved several structural results for Yangians of strange

Lie superalgebras of type $Q$, including:

(i) A description of an explicit algebraic basis for $\mathrm{Y}(\mathfrak{q}_N)$.

(ii) A description of the supercenter of $\mathrm{Y}(\mathfrak{q}_N)$.

In this dissertation, we extend to type $P$ many of the structural results for the type $Q$ case as in [Naz99]. Unlike our investigation into the orthosymplectic Yangians in Chapter 3, however, we will not study the representation theoretic aspects of $\mathrm{Y}(\mathfrak{p}_N)$. In fact, even though the defining relations of the Yangian $\mathrm{Y}(\mathfrak{q}_N)$ are relatively more agreeable when compared to its strange counterpart $\mathrm{Y}(\mathfrak{p}_N)$, the study into the representation theory for the Yangians of type $Q$ is still only at early development. Although Nazarov constructed functors between the representation categories of $\mathrm{Y}(\mathfrak{q}_N)$ and the *degenerate affine Sergeev algebras*, thereby presenting a wide array of finite-dimensional irreducible representations for the Yangian (see [Naz99, §5]), the classification of all finite-dimensional irreducible representations of $\mathrm{Y}(\mathfrak{q}_N)$ has only recently been completed when $N = 1$ (refer to [PS21]).

We now provide the main theorem of Chapter 4 and of this topic, which is the statement of a Poincaré-Birkhoff-Witt-type theorem for $\mathrm{Y}(\mathfrak{p}_N)$. We present the PBW-type theorem given as part (a) of Theorem VI which asserts that the associated graded superalgebra $\operatorname{gr}\mathrm{Y}(\mathfrak{p}_N)$ of the Yangian, with respect to a certain filtration, must be isomorphic to the universal enveloping superalgebra $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\vartheta})$:

**Theorem VI.** *The Yangian* $\mathrm{Y}(\mathfrak{p}_N)$ *has the following structural properties:*

(a) *There exists an involution* $\vartheta \in \mathrm{Aut}(\mathfrak{gl}_{N|N})$ *such that* $\mathfrak{p}_N = \mathfrak{gl}_{N|N}^{\vartheta}$ *and the non-trivial extension of* $\vartheta$ *to* $\mathfrak{gl}_{N|N}[z]$ *yields an* $\mathbb{N}$*-graded Hopf superalgebra isomorphism*

$$\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\vartheta}) \cong \operatorname{gr}\mathrm{Y}(\mathfrak{p}_N).$$

(b) *The supercenter* $\mathrm{ZY}(\mathfrak{p}_N)$ *of* $\mathrm{Y}(\mathfrak{p}_N)$ *is trivial:* $\mathbb{C}\cdot\mathbf{1}$.

(c) *There is a Hopf superalgebra embedding*

$$\iota\colon \mathfrak{U}(\mathfrak{p}_N) \hookrightarrow \mathrm{Y}(\mathfrak{p}_N).$$

The conglomerated results in Theorem VI can all be found in §4.3 as stated in Theorem 4.3.2, Corollary 4.3.4, and Proposition 4.3.5, respectively.

## 1.3 Twisted Super Yangians of Type AIII

### 1.3.1 Motivation and main results

Twisted Yangians serve as one of the main examples of quantum symmetric pairs of affine type, while at the same time form important instances of reflection algebras with additional symmetry and/or unitarity conditions. Namely, when $\mathfrak{g}$ is a finite-dimensional simple Lie algebra (or $\mathfrak{gl}_N$) and $\mathfrak{g}^\vartheta$ is the fixed-point Lie subalgebra under an involution $\vartheta \in \mathrm{Aut}(\mathfrak{g})$, one can associate to a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\vartheta)$ a certain left coideal subalgebra $\mathrm{Y}(\mathfrak{g}, \mathfrak{g}^\vartheta)^{tw}$ of $\mathrm{Y}(\mathfrak{g})$ called the *twisted Yangian* corresponding to the pair $(\mathfrak{g}, \mathfrak{g}^\vartheta)$. In particular, we refer to the tuple $(\mathrm{Y}(\mathfrak{g}), \mathrm{Y}(\mathfrak{g}, \mathfrak{g}^\vartheta)^{tw})$ as a *quantum symmetric pair*. These twisted Yangians have been shown to be an integral part of many models in mathematical physics, such as open spin chains, vertex models, and integrable systems with boundaries, whilst also playing a meaningful part in quantum field theory (see [Skl88, DMS01, Mac02, Mac05]).

There has been much work completed in regards to twisted Yangians associated to symmetric pairs of types A, B, C, and D, including their representation theories. The symmetric pairs most relevant to this work are those of type A which take the form

$$\mathrm{AI}\colon (\mathfrak{gl}_N, \mathfrak{so}_N), \quad \mathrm{AII}\colon (\mathfrak{gl}_N, \mathfrak{sp}_N), \quad \text{and} \quad \mathrm{AIII}\colon (\mathfrak{gl}_N, \mathfrak{gl}_p \oplus \mathfrak{gl}_{N-p}) \quad \text{for} \quad 0 \le p < N.$$

The twisted Yangians corresponding to the above symmetric pairs have been extensively studied, including the classification of their finite-dimensional irreducible representations (see [Mol92, Mol98, Mol07] for types AI & AII and [MR02] for type AIII). For the treatment of twisted Yangians corresponding to symmetric pairs of types B, C, and D, we refer the reader to the articles [GR16, GRW17, GRW19b].

This work concerns the development of twisted Yangians corresponding to the super-analogue of symmetric pairs of type AIII: the *symmetric superpairs*

$$\mathrm{AIII}\colon (\mathfrak{gl}_{M|N}, \mathfrak{gl}_{p|q} \oplus \mathfrak{gl}_{(M-p)|(N-q)}) \quad \text{for} \quad 0 \le p \le M,\ 0 \le q < N.$$

Indeed, for such indices $p, q$, one can realize $\mathfrak{gl}_{p|q} \oplus \mathfrak{gl}_{(M-p)|(N-q)}$ as a fixed-point Lie sub-superalgebra $\mathfrak{gl}_{M|N}^\vartheta$ for some involution $\vartheta \in \mathrm{Aut}(\mathfrak{gl}_{M|N})$. Similar to the previous subsection, one can extend $\vartheta$ in a non-trivial way to an involutive automorphism of the polynomial current Lie superalgebra $\mathfrak{gl}_{M|N}[z]$ via the assignment

$$\vartheta(f(z)) = \vartheta(f)(-z) \quad \text{for all} \quad f(z) \in \mathfrak{gl}_{M|N}[z].$$

The twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw}$ will be a particular left coideal sub-superalgebra of the Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N})$. Its relation to the symmetric superpair is realized by the fact that its parametrized twisted Yangian $\mathrm{Y}_{\hbar}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw} \subset \mathrm{Y}_{\hbar}(\mathfrak{gl}_{M|N})$ is a homogeneous superalgebra deformation of $\mathfrak{U}(\mathfrak{gl}_{M|N}[z]^{\vartheta})$:

$$\mathrm{Y}_{\hbar}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw} / \hbar\, \mathrm{Y}_{\hbar}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw} \cong \mathfrak{U}(\mathfrak{gl}_{M|N}[z]^{\vartheta}).$$

We shall now provide the main findings of Chapter 5 concerning these twisted super Yangians. The first main result comprises Theorem VII which acts as a Poincaré-Birkhoff-Witt-type theorem for $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw}$, given as Corollary 5.1.12.

**Theorem VII.** *The twisted Yangian* $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw}$ *is a filtered deformation of* $\mathfrak{U}(\mathfrak{gl}_{M|N}[z]^{\vartheta})$*, i.e., there exists an* $\mathbb{N}$*-graded superalgebra isomorphism*

$$\mathfrak{U}(\mathfrak{gl}_{M|N}[z]^{\vartheta}) \cong \operatorname{gr} \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw}.$$

In § 5.2, we develop a highest weight theory for studying the representations of $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw}$. In particular, the following theorem is an initial important result for addressing the the classification of all finite-dimensional irreducible representations of $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw}$:

**Theorem VIII.** *Every finite-dimensional irreducible representation $V$ of the twisted Yangian* $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{M|N}^{\vartheta})^{tw}$ *is a highest weight representation. The highest weight vector of $V$ is unique up to scalar multiple.*

We refer the reader to Theorem 5.2.3 for proof of Theorem VIII.

# Part One

## The Orthosymplectic Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$

# Chapter 2

# Yangians of Orthosymplectic Lie Superalgebras

In this chapter, we establish many structural properties for both the Yangians and extended Yangians of the orthosymplectic Lie superalgebras. The results proven here will be leveraged in Chapter 3, wherein we investigate their representation theories with the ultimate goal to classify their respective finite-dimensional irreducible representations.

We outline the chapter as follows. The first section §2.1 will serve as the preliminary component of both the chapter and the dissertation by recalling the definition of the Yangian of a finite-dimensional simple Lie algebra and introducing notation that will be standard across all chapters in this work. In §2.2, the definitions of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ and Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ are provided via the *RTT* realization. The primary result of the chapter resides in §2.3, where the PBW-type theorem for the Yangian is proven. The subsection §2.3.3 provides a comprehensive account of the Lie superbialgebra structure on $\mathfrak{osp}_{M|N}[z]$ and introduces the Yangian $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ via the Rees superalgebra formalism. In particular, there is a detailed explanation of how $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ serves as a homogeneous quantization of $\mathfrak{osp}_{M|N}[z]$. The final section §2.4 establishes many structural properties of the extended Yangian, including a tensor product decomposition, a PBW-type theorem, and the realization of the Yangian as a fixed-point subalgebra under a parametrized family of automorphisms of the extended Yangian.

## 2.1 Preliminaries

By convention, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{Z}$ is the set of all integers, $\mathbb{Z}^+$ denotes the set of positive integers, $\mathbb{C}$ is the field of complex numbers, $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ denotes the field of two elements. Let us fix our ground field to be $\mathbb{C}$. Unless otherwise stated, all linear algebraic notions are formulated with respect to this fixed ground field $\mathbb{C}$, i.e., vector space = $\mathbb{C}$-vector space, algebra = $\mathbb{C}$-algebra, linear map = $\mathbb{C}$-linear map, $\otimes = \otimes_{\mathbb{C}}$, etcetera.

### 2.1.1 The Yangian of a simple Lie algebra

Let $\mathfrak{g}$ denote a finite-dimensional complex simple Lie algebra equipped with a non-degenerate symmetric $\mathfrak{g}$-invariant bilinear form $(\cdot, \cdot)$ and fix an orthonormal basis $\{X_\lambda\}_{\lambda\in\Lambda}$ of $\mathfrak{g}$ with respect to such form, where $\Lambda$ is an index set of cardinality $\dim(\mathfrak{g})$. As defined in terms of its original $J$-presentation in [Dri85], the Yangian of $\mathfrak{g}$ is the following:

**Definition 2.1.1.** The *Yangian* $\mathrm{Y}(\mathfrak{g})$ is the unital associative $\mathbb{C}$-algebra generated by the elements $\{X, J(X)\}_{X\in\mathfrak{g}}$ subject to the following relations:

$$
\begin{aligned}
&[X,Y]_{\mathfrak{g}} = [X,Y], \quad J([X,Y]) = [J(X),Y],\\
&J(aX+bY) = aJ(X) + bJ(Y),\\
&[J(X),[J(Y),Z]] - [X,[J(Y),J(Z)]]\\
&\qquad\qquad\qquad = \sum_{\lambda,\mu,\upsilon\in\Lambda} \big([X,X_\lambda],[[Y,X_\upsilon],[Z,X_\upsilon]]\big)\{X_\lambda, X_\mu, X_\upsilon\},\\
&[[J(X),J(Y)],[Z,J(W)]] + [[J(Z),J(W)],[X,J(Y)]]\\
&\quad = \sum_{\lambda,\mu,\upsilon\in\Lambda} \Big(\big([X,X_\lambda],[[Y,X_\mu],[[Z,W],X_\upsilon]]\big)\\
&\qquad\qquad + \big([Z,X_\lambda],[[W,X_\mu],[[X,Y],X_\upsilon]]\big)\Big)\{X_\lambda, X_\mu, J(X_\upsilon)\},
\end{aligned}
$$

for all $W, X, Y, Z \in \mathfrak{g}$ and for all $a, b \in \mathbb{C}$, where

$$\{Z_1, Z_2, Z_3\} := \frac{1}{24}\sum_{\pi\in\mathfrak{S}_3} Z_{\pi(1)}Z_{\pi(2)}Z_{\pi(3)} \quad \text{for all} \quad Z_1, Z_2, Z_3 \in \mathrm{Y}(\mathfrak{g}).$$

We observe that the Yangian is $\mathbb{N}$-graded given by $\deg X = 0$ and $\deg J(X) = 1$ for elements $X \in \mathfrak{g}$. The Yangian $\mathrm{Y}(\mathfrak{g})$ admits at least three equivalent presentations, with the $J$-presentation being the first given above. In [Dri88], Drinfel'd discovered a second presentation of the Yangian, suitably called Drinfel'd's second (or current) realization, inspired by the Chevalley-Serre presentation of $\mathfrak{g}$. We will not focus on either the first or second presentation of the Yangian in the remainder of this work and instead forward to the third presentation which will dominate this dissertation: the $RTT$ realization.

Let us assume $M, N \in \mathbb{N}$ such that $N$ is even and $M+N \geq 1$. Further, define the sign $\theta_i = 1$ for $1 \leq i \leq M+\frac{N}{2}$ and $\theta_i = -1$ for $M+\frac{N}{2}+1 \leq i \leq M+N$. The *transposition* is the $\mathbb{C}$-linear map defined by

$$(-)^t\colon \operatorname{End}\mathbb{C}^{M+N} \to \operatorname{End}\mathbb{C}^{M+N}, \quad E_{ij} \mapsto E_{ij}^t := \theta_i\theta_j E_{\bar{\jmath}\bar{\imath}},$$

where $\bar{\imath} = M+N+1-i$ for the indices $1 \leq i \leq M+N$. The *permutation operator* in the space $\left(\operatorname{End}\mathbb{C}^{M+N}\right)^{\otimes 2}$ is given by

$$\mathsf{P} := \sum_{i,j=1}^{M+N} E_{ij} \otimes E_{ji} \in \left(\operatorname{End}\mathbb{C}^{M+N}\right)^{\otimes 2} \tag{2.1.1}$$

and further define the $\mathsf{Q}$ operator

$$\mathsf{Q} := (\mathrm{id} \otimes (-)^t)\mathsf{P} = \sum_{i,j=1}^{M+N} \theta_i\theta_j E_{ij} \otimes E_{\bar{\imath}\bar{\jmath}} \in \left(\operatorname{End}\mathbb{C}^{M+N}\right)^{\otimes 2}.$$

The $\mathsf{R}$-*matrix* $\mathsf{R}(u)$ is the rational function in the formal parameter $u$ taking coefficients in $\left(\operatorname{End}\mathbb{C}^{M+N}\right)^{\otimes 2}$ given by

$$\mathsf{R}(u) := \mathrm{id}^{\otimes 2} - \frac{\mathsf{P}}{u} + \frac{\mathsf{Q}}{u-\mathsf{k}} \in \left(\operatorname{End}\mathbb{C}^{M+N}\right)^{\otimes 2}(u), \tag{2.1.2}$$

where $\mathsf{k} = \mathsf{k}_{M,N} := (M+N-2\delta_{0N}+2\delta_{0M})/2$ and $\delta_{0D}$ is the Kronecker delta. When $M = 0$ or $N = 0$, it is known that the R-matrix (2.1.2) is a solution to the *quantum Yang-Baxter equation* (QYBE):

$$\mathsf{R}_{12}(u)\mathsf{R}_{13}(u+v)\mathsf{R}_{23}(v) = \mathsf{R}_{23}(v)\mathsf{R}_{13}(u+v)\mathsf{R}_{12}(u), \tag{2.1.3}$$

c.f. [ZZ79, KS82b, AAC+03]. For a description of the notation used in (2.1.3) and Definition 2.1.2 below, we refer the reader to subsection §2.1.3.

**Definition 2.1.2.** Let $\mathrm{X}_{M|N}$ be the unital associative $\mathbb{C}$-algebra on the generators $\{\mathsf{t}_{ij}^{(n)} \mid 1 \leq i,j \leq M+N,\ n \in \mathbb{Z}^+\}$ subject to the defining RTT-relation

$$\begin{aligned} \mathsf{R}(u-v)\mathsf{t}_1(u)\mathsf{t}_2(v) &= \mathsf{t}_2(v)\mathsf{t}_1(u)\mathsf{R}(u-v) \\ \text{in} \quad \left(\operatorname{End}\mathbb{C}^{M+N}\right)^{\otimes 2} &\otimes \mathrm{X}_{M|N}[\![u^{\pm 1}, v^{\pm 1}]\!], \end{aligned}$$

where $\mathsf{t}(u) := \sum_{i,j=1}^{M+N} E_{ij} \otimes \mathsf{t}_{ij}(u) \in \operatorname{End}(\mathbb{C}^{M+N}) \otimes \mathrm{X}_{M|N}[\![u^{-1}]\!]$ is the matrix consisting of the generating series $\mathsf{t}_{ij}(u) := \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} \mathsf{t}_{ij}^{(n)} u^{-n} \in \mathrm{X}_{M|N}[\![u^{-1}]\!]$, and $\mathsf{R}(u-v)$ is identified with $\mathsf{R}(u-v) \otimes \mathbf{1}$. Here, $\delta_{ij}$ denotes the Kronecker delta.

When $N = 0$, the algebra $\mathrm{X}_{M|0}$ is called the *extended Yangian* $\mathrm{X}(\mathfrak{so}_M)$ of the orthogonal Lie algebra $\mathfrak{so}_M$, whereas if $M = 0$, the algebra $\mathrm{X}_{0|N}$ is called the *extended Yangian* $\mathrm{X}(\mathfrak{sp}_N)$ of the symplectic Lie algebra $\mathfrak{sp}_N$.

Setting $\mathsf{T}^t(u+\mathsf{k}) := ((-)^t \otimes \mathrm{id})\mathsf{T}(u+\mathsf{k})$, we consider the matrix $\mathsf{Z}(u) := \mathsf{T}^t(u+\mathsf{k})\mathsf{T}(u)$ and the series $z(u) \in \mathbf{1} + u^{-1}\mathrm{X}_{M|N}[\![u^{-1}]\!]$ defined by $\mathrm{id} \otimes z(u) = \mathsf{Z}(u)$. Allowing $(z(u) - \mathbf{1})$ to denote the two-sided ideal of $\mathrm{X}_{M|N}$ generated by the coefficients of $z(u) - \mathbf{1}$, we arrive at another definition of the Yangian for the Lie algebras $\mathfrak{so}_M$ and $\mathfrak{sp}_N$:

**Definition 2.1.3.** Let $\mathrm{Y}_{M|N}$ be the quotient of $\mathrm{X}_{M|N}$ by the two-sided ideal $(z(u) - \mathbf{1})$:

$$\mathrm{Y}_{M|N} := \mathrm{X}_{M|N} / (\mathsf{Z}(u) - \mathbf{1}). \tag{2.1.4}$$

When $N = 0$, the quotient $\mathrm{Y}_{M|0}$ is called the *Yangian* $\mathrm{Y}(\mathfrak{so}_M)$ of $\mathfrak{so}_M$. Accordingly, if $M = 0$, the quotient $\mathrm{Y}_{0|N}$ is called the *Yangian* $\mathrm{Y}(\mathfrak{sp}_N)$ of $\mathfrak{sp}_N$.

We refer the reader to [GRW19a] for a detailed exposition on the equivalence of the three aforementioned presentations for the orthogonal and symplectic Yangians.

### 2.1.2 The gradation index and orthosymplectic Lie superalgebra

If $V$ denotes a vector space with an ordered basis $\{x_i\}_{i=1}^{D}$, a natural way to equip $V$ with a $\mathbb{Z}_2$-grading is by specifying its first $d$ many basis vectors, with $d \leq D$, to be even whilst setting the remaining basis vectors to be odd. We note the more general case when one may set *any* $d$ many basis vectors to be even, which prompts the following definition:

**Definition 2.1.4.** Fix two integers $d \in \mathbb{N}$, $D \in \mathbb{Z}^+$ such that $d \leq D$. For a subset $\mathbf{d} \subseteq \{1, 2, \ldots, D\}$ of cardinality $d$, we introduce the *gradation index*

$$[\,\cdot\,]_{\mathbf{d}} \colon \{1, 2, \ldots, D\} \to \mathbb{Z}_2 \tag{2.1.5}$$

given by $[i]_{\mathbf{d}} = \bar{0}$ for $i \in \mathbf{d}$ and $[i]_{\mathbf{d}} = \bar{1}$ for $i \in \mathbf{d}' = \{1, 2, \ldots, D\} \setminus \mathbf{d}$. When $\mathbf{d} = \{1, 2, \ldots, d\}$, we set $[\,\cdot\,] = [\,\cdot\,]_{\mathbf{d}}$.

We will primarily be working with super vector spaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$ that are graded with respect to the gradation index (2.1.5), but we shall also denote the gradation of homogeneous elements in $V$ with the similar notation: $[\,\cdot\,] \colon V_{\bar{0}} \sqcup V_{\bar{1}} \to \mathbb{Z}_2, v \mapsto [v]$, where $[v] = \gamma \in \mathbb{Z}_2$ if $v \in V_\gamma$. For a super vector space $V$, elements in $V_{\bar{0}}$ are said to be *even* and elements in $V_{\bar{1}}$ are said to be *odd*.

The prototypical vector space we will use that is graded with respect to the gradation index (2.1.5) is the space $\mathbb{C}^{M+N}$, where $M, N \in \mathbb{N}$ such that $M+N \geq 1$. We let the standard basis be given by $\mathbf{B} = \{e_i\}_{i=1}^{M+N}$ and denote $\mathbb{C}_{\mathbf{d}}^{M|N}$ to be such vector space equipped with the $\mathbb{Z}_2$-grading given by $[e_i] := [i]_{\mathbf{d}}$ for $1 \leq i \leq M+N$, where $d = M$ and $D = M+N$ as in Definition 2.1.4. When $\mathbf{d} = \{1, 2, \ldots, M\}$, we set $\mathbb{C}^{M|N} = \mathbb{C}_{\mathbf{d}}^{M|N}$.

Setting $V = \mathbb{C}_{\mathbf{d}}^{M|N}$, the space of $\mathbb{C}$-linear maps $V \to V$, denoted $\operatorname{End} V$, carries a natural $\mathbb{Z}_2$-grading via the assignment $(\operatorname{End} V)_\gamma = \{\varphi \in \operatorname{End} V \mid \varphi(V_\eta) \subseteq V_{\eta+\gamma},\ \eta \in \mathbb{Z}_2\}$. In fact, such grading is provided by $[E_{ij}] := [i]_{\mathbf{d}} + [j]_{\mathbf{d}}$, where $\{E_{ij}\}_{i,j=1}^{D}$ is the collection of the matrix units of $\operatorname{End} V$ with respect to the basis $\mathbf{B}$.

Again assume $M, N \in \mathbb{N}$ such that $M+N \geq 1$, $N$ is even, and set $d = M$, $D = M+N$ as in Definition 2.1.4. Further, consider $\mathbf{d}$ and $\mathbf{d}'$ as ordered sets with respect to the canonical ordering on $\mathbb{N}$, so that $\mathbf{d}[j]$ and $\mathbf{d}'[j]$ denotes the $j^{th}$ elements in these sets and $\mathbf{d}'_{N/2}$ denotes the first $\frac{N}{2}$ integers in $\mathbf{d}'$. For each $1 \leq i \leq M+N$, define the sign

$$\theta_i^{\mathbf{d}} := \begin{cases} 1 & \text{if} \quad i \in \mathbf{d} \cup \mathbf{d}'_{N/2}, \\ -1 & \text{if} \quad i \in \mathbf{d}' \setminus \mathbf{d}'_{N/2}, \end{cases}$$

and define the conjugate index $\bar{\imath}^{\mathbf{d}}$ as

$$\bar{\imath}^{\mathbf{d}} := \begin{cases} \mathbf{d}[M+1-j] & \text{if} \quad i = \mathbf{d}[j] \text{ for some } 1 \le j \le M, \\ \mathbf{d}'[N+1-j] & \text{if} \quad i = \mathbf{d}'[j] \text{ for some } 1 \le j \le N. \end{cases}$$

We shall see that the definitions of the Yangian depend on the selection of the set $\mathbf{d}$, which will be more relevant in Chapter 3. For the remainder of this chapter, however, we shall assume $\mathbf{d} = \{1, 2, \dots, M\}$ unless otherwise specified. Accordingly, we denote $\theta_i := \theta_i^{\mathbf{d}}$ and $\bar{\imath} := \bar{\imath}^{\mathbf{d}}$ in this case, which means such symbols are given by

$$\theta_i = \begin{cases} 1 & \text{if} \quad 1 \le i \le M + \frac{N}{2}, \\ -1 & \text{if} \quad M + \frac{N}{2} + 1 \le i \le M + N, \end{cases}$$

and

$$\bar{\imath} = \begin{cases} M+1-i & \text{if} \quad 1 \le i \le M, \\ 2M+N+1-i & \text{if} \quad M+1 \le i \le M+N. \end{cases}$$

Generally, spaces in this work will be regarded as an object in the symmetric monoidal category of super vector spaces over $\mathbb{C}$, denoted $\mathrm{sVect}_{\mathbb{C}}$, which is equipped with the *super-braiding* $\sigma$. As such, given any two objects $V$ and $W$ in $\mathrm{sVect}_{\mathbb{C}}$, there is an isomorphism $\sigma_{V,W} \colon V \otimes W \to W \otimes V$, $v \otimes w \mapsto (-1)^{[v][w]} w \otimes v$ on homogeneous elements $v \in V$ and $w \in W$. We note that it will be common to drop the subscripts on the super-braiding $\sigma_{V,W}$ throughout this dissertation.

For homogeneous linear maps $\varphi \in \operatorname{End} V$ and $\psi \in \operatorname{End} W$, their (super) tensor product is the homogeneous linear map in $\operatorname{End}(V \otimes W)$, denoted $\varphi \otimes \psi$, given by

$$\varphi \otimes \psi \colon V \otimes W \to V \otimes W, \quad v \otimes w \mapsto (-1)^{[\psi][v]} \varphi(v) \otimes \psi(w).$$

on homogeneous vectors $v \in V$, $w \in W$. Note that when $\varphi$ and $\psi$ are even (or just $\psi$), then their (super) tensor product is simply the traditional tensor product of linear maps. For instance, the operator $E_{ij} \otimes E_{kl}$ in $\left(\operatorname{End} \mathbb{C}^{M|N}\right)^{\otimes 2} \cong \operatorname{End}(\mathbb{C}^{M|N} \otimes \mathbb{C}^{M|N})$ acts on basis elements $e_a \otimes e_b$ via the formula

$$(E_{ij} \otimes E_{kl}) \, (e_a \otimes e_b) = \delta_{ja} \delta_{lb} (-1)^{([k]+[l])[a]} e_i \otimes e_k.$$

If we further suppose that $V$ and $W$ are algebras in $\mathrm{sVect}_{\mathbb{C}}$ with multiplication maps

$\mu_V : V \otimes V \to V$ and $\mu_W : W \otimes W \to W$, then multiplication in $V \otimes W$ is defined by the composition $(\mu_V \otimes \mu_W) \circ (\mathrm{id}_V \otimes \sigma \otimes \mathrm{id}_W)$. Explicitly, this multiplication is given by $(v_1 \otimes w_1)(v_2 \otimes w_2) = (-1)^{[w_1][v_2]} v_1 v_2 \otimes w_1 w_2$ on homogeneous elements. When $V$ is associative, we shall let $\mathrm{Lie}(V)$ denote the Lie superalgebra structure on $V$ given by the super-commutator $[v_1, v_2] = v_1 v_2 - (-1)^{[v_1][v_2]} v_2 v_1$ for homogeneous elements $v_1, v_2 \in V$.

Given a superalgebra $\mathcal{A}$, we shall let $\mathrm{Mat}_{M|N}(\mathcal{A})$ denote the collection of supermatrices over $\mathcal{A}$ with dimension $M|N$. As a set, $\mathrm{Mat}_{M|N}(\mathcal{A})$ coincides with $\mathrm{Mat}_{M+N}(\mathcal{A})$ but each supermatrix $A \in \mathrm{Mat}_{M|N}(\mathcal{A})$ of $\mathbb{Z}_2$-degree $[A] = \gamma$ is a $2 \times 2$ block matrix

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \tag{2.1.6}$$

such that $A_{00} \in \mathrm{Mat}_M(\mathcal{A}_\gamma)$, $A_{01} \in \mathrm{Mat}_{M\times N}(\mathcal{A}_{\gamma+\bar{1}})$, $A_{10} \in \mathrm{Mat}_{N\times M}(\mathcal{A}_{\gamma+\bar{1}})$, and $A_{11} \in \mathrm{Mat}_N(\mathcal{A}_\gamma)$. Via traditional matrix multiplication, the collection $\mathrm{Mat}_{M|N}(\mathcal{A})$ forms a superalgebra structure over $\mathcal{A}$.

When $\mathcal{A}$ is not super, one can naturally identify the algebra $\mathrm{End}(\mathbb{C}^{M+N}) \otimes \mathcal{A}$ with $\mathrm{Mat}_{M+N}(\mathcal{A})$ so that multiplication in $\mathrm{End}(\mathbb{C}^{M+N}) \otimes \mathcal{A}$ may be simply regarded as matrix multiplication. However, when $\mathcal{A}$ is super, one invariably encounters signs occurring with multiplication in $\mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathcal{A}$ that does not occur with ordinary (super) matrix multiplication in $\mathrm{Mat}_{M|N}(\mathcal{A})$. We therefore observe there is an algebra isomorphism

$$\big(\mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathcal{A}\big)_{\bar{0}} \xrightarrow{\sim} \mathrm{Mat}_{M|N}(\mathcal{A})_{\bar{0}}, \quad \textstyle\sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes A_{ij} \mapsto (A_{ij})_{i,j=1}^{M+N},$$

where the elements $A_{ij} \in \mathcal{A}$ are homogeneous of degree $[A_{ij}] = [E_{ij}] = [i]+[j]$.

For the remainder of this section, we shall only consider the case when $\mathcal{A}$ is the base field $\mathbb{C} = \mathbb{C}^{1|0}$ with the trivial $\mathbb{Z}_2$-grading by deeming all elements as even. The *super-transpose* is the $\mathbb{C}$-linear map defined by

$$(-)^{st} \colon \mathrm{End}\,\mathbb{C}^{M|N} \to \mathrm{End}\,\mathbb{C}^{M|N}, \quad E_{ij} \mapsto E_{ij}^{st} := (-1)^{[i][j]+[i]} E_{ji} \tag{2.1.7}$$

and is furthermore a superalgebra anti-automorphism: $(A_1 A_2)^{st} = (-1)^{[A_1][A_2]} A_2^{st} A_1^{st}$ for any homogeneous elements $A_1, A_2 \in \mathrm{End}\,\mathbb{C}^{M|N}$. If $(-)'$ denotes the conventional matrix transpose, then via the superalgebra identification $\mathrm{End}\,\mathbb{C}^{M|N} \cong \mathrm{Mat}_{M|N}(\mathbb{C})$, the super-transpose $A^{st}$ of $A = (A_{ij})_{i,j=0}^{1} \in \mathrm{Mat}_{M|N}(\mathbb{C})$ is provided by

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}^{st} = \begin{pmatrix} A_{00}' & A_{10}' \\ -A_{01}' & A_{11}' \end{pmatrix}.$$

The space $\mathrm{End}\,\mathbb{C}^{M|N}$ is a unital associative superalgebra and therefore carries the structure of a Lie superalgebra given by the super-commutator

$$[E_{ij}, E_{kl}] := \delta_{jk} E_{il} - \delta_{il} (-1)^{([i]+[j])([k]+[l])} E_{kj}$$

for indices $1 \leq i, j \leq M+N$, where $\delta_{ij}$ is the Kronecker delta. When equipped with the above Lie superalgebra structure, we shall denote the space $\mathrm{End}\,\mathbb{C}^{M|N}$ as $\mathfrak{gl}_{M|N} = \mathfrak{gl}(\mathbb{C}^{M|N})$ and call it the *general Lie superalgebra.*

**Definition 2.1.5.** Assume $b\colon \mathbb{C}^{M|N} \times \mathbb{C}^{M|N} \to \mathbb{C}$ is an even, super-symmetric, non-degenerate $\mathbb{C}$-bilinear form; hence, $N$ is necessarily even. The *orthosymplectic Lie superalgebra* $\mathfrak{osp}_{M|N} = \mathfrak{osp}_{M|N}(\mathbb{C}^{M|N}, b)$ is defined as the Lie sub-superalgebra of $\mathfrak{gl}_{M|N}$ preserving such bilinear form $b$.

That is, $\mathfrak{osp}_{M|N}$ is the Lie sub-superalgebra generated by homogeneous elements $\varphi \in \mathfrak{gl}_{M|N}$ satisfying the relation $b(\varphi(v), w) + (-1)^{[\varphi][v]} b(v, \varphi(w)) = 0$ for all homogeneous vectors $v, w \in \mathbb{C}^{M|N}$. The associated matrix of $b$ with respect to the standard basis $\{e_i\}_{i=1}^{M+N}$ is given by $B = \big(b(e_i, e_j)\big)_{i,j=1}^{M+N}$, which necessarily has the form

$$B = \begin{pmatrix} G & 0 \\ 0 & J \end{pmatrix}$$

where $G \in \mathrm{Mat}_M(\mathbb{C})$ and $J \in \mathrm{Mat}_N(\mathbb{C})$ are invertible matrices satisfying $G' = G$ and $J' = -J$. As the definition of $\mathfrak{osp}_{M|N}$ is independent of the selection of such a bilinear form, we may assume

$$G = \big(\delta_{i\bar{j}}\big)_{i,j=1}^{M} \quad \text{and} \quad J = \big(\theta_j \delta_{i\bar{j}}\big)_{i,j=M+1}^{M+N}, \quad \text{so} \quad B = \big(\theta_j \delta_{i\bar{j}}\big)_{i,j=1}^{M+N}. \tag{2.1.8}$$

Regarding $A \in \mathfrak{gl}_{M|N}$ in its matrix form (2.1.6), one therefore has $A \in \mathfrak{osp}_{M|N}$ if and only if $A^{st}B + BA = 0$, i.e., if $A_{00} \in \mathfrak{so}_M$, $A_{11} \in \mathfrak{sp}_N$, and $A_{10}' J + G A_{01} = 0$.

Given the *super-trace* $\mathrm{str}\colon \mathfrak{gl}_{M|N} \to \mathbb{C}$, $A \mapsto \mathrm{tr}(A_{00}) - \mathrm{tr}(A_{11})$, where $A$ is of the form (2.1.6), *the special linear Lie superalgebra* $\mathfrak{sl}_{M|N}$ is the Lie sub-superalgebra of $\mathfrak{gl}_{M|N}$ defined by the set $\{X \in \mathfrak{gl}_{M|N} \mid \mathrm{str}(X) = 0\}$. Since $\mathfrak{so}_M \subset \mathfrak{sl}_M$ and $\mathfrak{sp}_N \subset \mathfrak{sl}_N$, it follows that $\mathfrak{osp}_{M|N} \subset \mathfrak{sl}_{M|N}$.

We will now show that one may also regard $\mathfrak{osp}_{M|N}$ as a certain fixed-point Lie sub-superalgebra of $\mathfrak{sl}_{M|N}$ under some involution $\vartheta$. It is this realization of the orthosymplectic Lie superalgebras that will be utilized more prominently in this chapter, and subsequently, Chapter 3. To this end, we introduce the *super-transposition* as the $\mathbb{C}$-linear map defined by

$$(-)^t\colon \operatorname{End}\mathbb{C}^{M|N} \to \operatorname{End}\mathbb{C}^{M|N}, \quad E_{ij} \mapsto E_{ij}^t := (-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{\imath}}. \tag{2.1.9}$$

Similar to the super-transpose (2.1.7), the super-transposition is a superalgebra anti-automorphism: $(A_1A_2)^t = (-1)^{[A_1][A_2]}A_2^tA_1^t$ for homogeneous maps $A_1, A_2 \in \operatorname{End}\mathbb{C}^{M|N}$. However, we note that the super-transposition is in fact an involution, unlike the super-transpose which is of order 4. Moreover, the super-transposition and the super-transpose commute: $(-)^t \circ (-)^{st} = (-)^{st} \circ (-)^t$; and given any index $1 \le k \le m$, it will be convention throughout this work to let $(-)^{t_k}$ denote the map

$$\mathrm{id}^{\otimes(k-1)} \otimes (-)^t \otimes \mathrm{id}^{\otimes(m-k)} \in \operatorname{End}\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes m}.$$

Via the super-transposition, there is an involutive automorphism $\vartheta \in \mathrm{Aut}(\mathfrak{sl}_{M|N})$ defined by

$$\vartheta := -(-)^t\colon \mathfrak{sl}_{M|N} \to \mathfrak{sl}_{M|N}, \quad X \mapsto -X^t,$$

and one can show that the orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|N}$ coincides with the fixed-point sub-superalgebra $\mathfrak{sl}_{M|N}^{\vartheta}$ of $\mathfrak{sl}_{M|N}$ under such involution $\vartheta$. In particular, the Lie superalgebra $\mathfrak{osp}_{M|N}$ is generated by the operators

$$F_{ij} := E_{ij} + \vartheta(E_{ij}) = E_{ij} - (-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{\imath}} \in \mathfrak{sl}_{M|N} \tag{2.1.10}$$

with indices $1 \le i, j \le M+N$, subject only to the relations

$$\begin{aligned}\left[F_{ij}, F_{kl}\right] = \delta_{jk}F_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}F_{kj} \\ - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}l} + \delta_{\bar{j}l}(-1)^{([i]+[j])[k]}\theta_{\bar{\imath}}\theta_{\bar{j}}F_{k\bar{\imath}}\end{aligned} \tag{2.1.11}$$

and

$$F_{ij} + (-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}\bar{\imath}} = 0. \tag{2.1.12}$$

### 2.1.3 Mapping notation

Let $W$ be an arbitrary super vector space and let $V$ be a finite-dimensional super vector space of dimension $d$ with a fixed basis $\{b_1, \ldots, b_d\}$, where $\{E_{ij}\}_{i,j=1}^d$ denotes the matrix units of $\operatorname{End} V$ with respect to this basis. In this work, we will often represent objects in $(\operatorname{End} V) \otimes W$ in the larger ambient space $(\operatorname{End} V)^{\otimes m} \otimes W$ for some integer $m \geq 2$. For this purpose, any index $1 \leq k \leq m$ will determine a morphism of super vector spaces

$$\begin{aligned}\varphi_k \colon (\operatorname{End} V) \otimes W &\to (\operatorname{End} V)^{\otimes m} \otimes W \\ \psi \otimes w &\mapsto \mathrm{id}^{\otimes (k-1)} \otimes \psi \otimes \mathrm{id}^{\otimes (m-k)} \otimes w,\end{aligned}$$

and set $X_k = \varphi_k(X)$ for $X \in (\operatorname{End} V) \otimes W$. Explicitly, if $X = \sum_{i,j=1}^d E_{ij} \otimes w_{ij}$, then

$$X_k = \sum_{i,j=1}^{d} \mathrm{id}^{\otimes (k-1)} \otimes E_{ij} \otimes \mathrm{id}^{\otimes (m-k)} \otimes w_{ij} \in (\operatorname{End} V)^{\otimes m} \otimes W.$$

If $X = X(u)$ depends on some formal parameter $u$, we shall write $X_k(u)$ instead of $X(u)_k$ for the element $\varphi_k(X(u))$.

Generalizing of the above map when $W = \mathcal{A}$ is a superalgebra with unit $\mathbf{1}$, we will also aim to represent objects in $(\operatorname{End} V) \otimes \mathcal{A}$ in the larger space $(\operatorname{End} V)^{\otimes m} \otimes \mathcal{A}^{\otimes n}$ for some integers $m, n \in \mathbb{Z}^+$. For this, any indices $1 \leq k \leq m$ and $1 \leq l \leq n$ determine a morphism of superalgebras

$$\begin{aligned}\varphi_{k[l]} \colon (\operatorname{End} V) \otimes \mathcal{A} &\to (\operatorname{End} V)^{\otimes m} \otimes \mathcal{A}^{\otimes n} \\ \psi \otimes a &\mapsto \mathrm{id}^{\otimes (k-1)} \otimes \psi \otimes \mathrm{id}^{\otimes (m-k)} \otimes \mathbf{1}^{\otimes (l-1)} \otimes a \otimes \mathbf{1}^{\otimes (n-l)},\end{aligned}$$

and set $X_{k[l]} = \varphi_{k[l]}(X)$ for an element $X \in (\operatorname{End} V) \otimes \mathcal{A}$. Explicitly, if we express $X$ as the sum $\sum_{i,j=1}^d E_{ij} \otimes a_{ij}$, then

$$X_{k[l]} = \sum_{i,j=1}^{d} \mathrm{id}^{\otimes (k-1)} \otimes E_{ij} \otimes \mathrm{id}^{\otimes (m-k)} \otimes \mathbf{1}^{\otimes (l-1)} \otimes a_{ij} \otimes \mathbf{1}^{\otimes (n-l)} \in (\operatorname{End} V)^{\otimes m} \otimes \mathcal{A}^{\otimes n}.$$

If $k = 1$ we shall abbreviate $X_{1[l]}$ by $X_{[l]}$ and if $l = 1$ we shall abbreviate $X_{k[1]}$ by $X_k$ just as above. When $X = X(u)$ depends on some formal parameter $u$, we shall write $X_{k[l]}(u)$ instead of $X(u)_{k[l]}$ for the element $\varphi_{k[l]}(X(u))$.

Analogously, we will like to express elements of $\mathcal{A}\otimes\mathcal{A}$ within $\mathcal{A}^{\otimes m}$ for some integer $m\geq 3$. Any pair of indices $1\leq k<l\leq m$ will determine a morphism of superalgebras

$$\varphi_{kl}\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}^{\otimes m},\quad a\otimes b\mapsto \mathbf{1}^{\otimes(k-1)}\otimes a\otimes\mathbf{1}^{\otimes(l-k-1)}\otimes b\otimes\mathbf{1}^{\otimes(m-l)},$$

and set $X_{kl}=\varphi_{kl}(X)$ for an element $X\in\mathcal{A}\otimes\mathcal{A}$. Explicitly, if $X=\sum_{i=1}^{r}a_i\otimes b_i$, then

$$X_{kl}=\sum_{i=1}^{r}\mathbf{1}^{\otimes(k-1)}\otimes a_i\otimes\mathbf{1}^{\otimes(l-k-1)}\otimes b_i\otimes\mathbf{1}^{\otimes(m-l)}.$$

Again, when $X=X(u)$ depends on some formal parameter $u$, then we write $X_{kl}(u)$ instead of $X(u)_{kl}$ for the element $\varphi_{kl}(X(u))$.

In Yangian theory, it is convention to use formal power series to define maps between spaces since at least one of these spaces is usually (countably) infinite-dimensional and such notation offers the advantage of brevity. In particular, if $W_1$ and $W_2$ are super vector spaces or superalgebras with series $A(u)=\sum_{n=0}^{\infty}A_nu^{-n}\in W_1[\![u^{-1}]\!]$ and $B(u)=\sum_{n=0}^{\infty}B_nu^{-n}\in W_2[\![u^{-1}]\!]$, then we write

$$\varphi\colon A(u)\mapsto B(u)$$

to mean the map $\varphi(A_n)=B_n$ for all $n\in\mathbb{N}$. Typically, $W_1$ will be a superalgebra with generating set $\{A_n\}_{n=0}^{\infty}$, so provided the coefficients of $B(u)$ satisfy the necessary conditions, then $\varphi$ will define a morphism $W_1\to W_2$.

As we will see, the more relevant setup is the following. Supposing that $V$ is a super vector space of dimension $D\in\mathbb{Z}^+$ graded with respect to the gradation index (2.1.5), we consider the matrices $A(u)=\sum_{i,j=1}^{D}(-1)^{[i][j]+[j]}E_{ij}\otimes A_{ij}(u)\in\mathrm{End}(V)\otimes W_1[\![u^{-1}]\!]$ and $B(u)=\sum_{i,j=1}^{D}(-1)^{[i][j]+[j]}E_{ij}\otimes B_{ij}(u)\in\mathrm{End}(V)\otimes W_2[\![u^{-1}]\!]$ consisting of formal power series $A_{ij}(u)=\delta_{ij}\mathbf{1}+\sum_{n=1}^{\infty}A_{ij}^{(n)}u^{-n}$ and $B_{ij}(u)=\delta_{ij}\mathbf{1}+\sum_{n=1}^{\infty}B_{ij}^{(n)}u^{-n}$. One then writes

$$\varphi\colon A(u)\mapsto B(u)$$

to mean the map $\varphi(A_{ij}^{(n)})=B_{ij}^{(n)}$ for all $1\leq i,j\leq D$ and $n\in\mathbb{Z}^+$. When $W_1$ is a superalgebra with the generating set $\{A_{ij}^{(n)}\mid 1\leq i,j\leq D,\ n\in\mathbb{Z}^+\}$ and the coefficients of $B_{ij}(u)$, $1\leq i,j\leq D$, satisfy the necessary conditions, then $\varphi$ will define a morphism $W_1\to W_2$.

Lastly, assume $\mathcal{A}$ is a superalgebra generated by $\{A_{ij}^{(n)} \mid 1 \le i,j \le D,\ n \in \mathbb{Z}^+\}$. When describing the action of $\mathcal{A}$ on a vector $\xi$ in a representation $V$, it will be common to set $A_{ij}(u) = \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} A_{ij}^{(n)} u^{-n} \in \mathcal{A}[\![u^{-1}]\!]$ and $v_{ij}(u) = \sum_{n=1}^{\infty} v_{ij}^{(n)} u^{-n} \in V[\![u^{-1}]\!]$, so one can write

$$A_{ij}(u)\xi = \delta_{ij}\xi + v_{ij}(u)$$

to mean the action of $\mathcal{A}$ on $\xi \in V$ given by $A_{ij}^{(n)}\xi = v_{ij}^{(n)}$ for all $1 \le i,j \le D$ and $n \in \mathbb{Z}^+$.

## 2.2 Orthosymplectic Yangians

The first definition of the *Yangian* $\mathrm{Y}(\mathfrak{osp}_{M|N})$ for the orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|N}$ was given in [AAC+03, §3] via the *RTT* realization. In this section, we will recall such definition of the Yangian as a certain quotient of the *extended Yangian* $\mathrm{X}(\mathfrak{osp}_{M|N})$, which appeared also in the same article.

The following constructions will yield isomorphic presentations of the Yangians, and extended Yangians, of the orthogonal Lie algebra $\mathfrak{so}_M$ when $N = 0$, and the symplectic Lie algebra $\mathfrak{sp}_N$ when $M = 0$, whose *RTT* presentations were thoroughly examined in the paper [AMR06].

### 2.2.1 Extended orthosymplectic Yangians

As one can infer from §2.1.1, the *RTT* realization for Yangians of the orthosymplectic Lie superalgebras will rely on a solution to the super-analogue of the QYBE (2.1.3), so we accordingly construct such an $R$-matrix here. To start, the *super permutation operator* in $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$ is given by

$$P := \sum_{i,j=1}^{M+N} (-1)^{[j]} E_{ij} \otimes E_{ji} \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}. \tag{2.2.1}$$

Further, we define $Q \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$ as the transposed operator

$$Q := P^{t_1} = P^{t_2} = \sum_{i,j=1}^{M+N} (-1)^{[i][j]} \theta_i \theta_j E_{ij} \otimes E_{\bar{\imath}\bar{\jmath}} \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}, \tag{2.2.2}$$

where $(-)^t$ is the involution (2.1.9). The *R-matrix* $R(u)$ is the rational function in the formal parameter $u$ taking coefficients in $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$ given by

$$R(u) := \mathrm{id}^{\otimes 2} - \frac{P}{u} + \frac{Q}{u-\kappa} \in \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}(u), \quad \kappa = \kappa_{M,N} := \tfrac{M-N-2}{2}. \tag{2.2.3}$$

It is known that the $R$-matrix (2.2.3) is a solution to the *super quantum Yang-Baxter equation* (SQYBE):

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u), \tag{2.2.4}$$

c.f. [ZZ79, KS82b, AAC+03]. Moreover, the array of equalities

$$P^2 = \mathrm{id}^{\otimes 2}, \quad PQ = QP = Q, \quad \text{and} \quad Q^2 = (M-N)Q,$$

infer that the $R$-matrix $R(u)$ satisfies the properties

$$R^{t_1}(u+\kappa) = R^{t_2}(u+\kappa) = R(-u), \tag{2.2.5}$$

$$R(u)R(-u) = \left(1 - \frac{1}{u^2}\right)\mathrm{id}^{\otimes 2}, \tag{2.2.6}$$

known as *crossing symmetry* and *unitarity*, respectively.

A particular consequence of crossing symmetry is that the $R$-matrix is invariant under the map $(-)^t \otimes (-)^t$, which will be utilized later in the subsection. Using the $R$-matrix (2.2.3), we can introduce the definition of the extended Yangian:

**Definition 2.2.1.** The *extended Yangian* $\mathrm{X}(\mathfrak{osp}_{M|N})$ of $\mathfrak{osp}_{M|N}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{T_{ij}^{(n)} \mid 1 \le i,j \le M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\left[T_{ij}^{(n)}\right] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the defining *RTT-relation*

$$\begin{gathered} R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \\ \text{in} \quad \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{\pm 1}, v^{\pm 1}]\!], \end{gathered} \tag{2.2.7}$$

where $T(u) := \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes T_{ij}(u) \in \operatorname{End}(\mathbb{C}^{M|N}) \otimes \mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ is the matrix consisting of the series $T_{ij}(u) := \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} T_{ij}^{(n)} u^{-n} \in \mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ for indices $1 \le i,j \le M+N$, and $R(u-v)$ is identified with $R(u-v) \otimes \mathbf{1}$.

In terms of formal power series, the $RTT$-relation (2.2.7) equivalently takes the form

$$\begin{aligned}\big[T_{ij}(u),T_{kl}(v)\big] &= \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}(u)T_{il}(v)-T_{kj}(v)T_{il}(u)\Big)\\ &\quad -\frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v)\\ &\qquad -\delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg)\end{aligned} \tag{2.2.8}$$

for all $1 \le i,j,k,l \le M+N$, where the above equality may be regarded as one in the extension $\mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{\pm1},v^{\pm1}]\!]$ and $[\,\cdot\,,\cdot\,]$ is understood as the super-bracket

$$\big[T_{ij}(u),T_{kl}(v)\big] = T_{ij}(u)T_{kl}(v) - (-1)^{([i]+[j])([k]+[l])}T_{kl}(v)T_{ij}(u).$$

**Remark 2.2.2.** Definition 2.2.1 of $\mathrm{X}(\mathfrak{osp}_{M|N})$ inherently relies on the selection of the set $\mathbf{d}$ for the gradation index (2.1.5). Suppose that $\mathrm{X}^{\mathbf{d}_1}(\mathfrak{osp}_{M|N})$ and $\mathrm{X}^{\mathbf{d}_2}(\mathfrak{osp}_{M|N})$ denote two definitions of the extended Yangian in terms of two different sets $\mathbf{d}_1$ and $\mathbf{d}_2$ as in Definition 2.1.4; accordingly, we denote the generating series for each of these definitions as $T_{ij}^{\mathbf{d}_1}(u)$ and $T_{ij}^{\mathbf{d}_2}(u)$, respectively. if $\mathfrak{S}_{M+N}$ denotes the symmetric group on the symbols $\{1,2,\dots,M+N\}$ and the bijection $\sigma\in\mathfrak{S}_{M+N}$ satisfies $[i]_{\mathbf{d}_1}=[\sigma(i)]_{\mathbf{d}_2}$, $\theta_i^{\mathbf{d}_1}=\theta_{\sigma(i)}^{\mathbf{d}_2}$, and $\sigma(\bar{\imath}^{\mathbf{d}_1})=\overline{\sigma(i)}^{\mathbf{d}_2}$, then

$$\mathrm{X}^{\mathbf{d}_1}(\mathfrak{osp}_{M|N}) \xrightarrow{\sim} \mathrm{X}^{\mathbf{d}_2}(\mathfrak{osp}_{M|N}), \quad T_{ij}^{\mathbf{d}_1}(u)\mapsto T_{\sigma(i)\sigma(j)}^{\mathbf{d}_2}(u)$$

is an isomorphism of superalgebras.

**Remark 2.2.3.** When $N=0$, the non-super permutation operator (2.1.1) and super permutation operator (2.2.1) coincide: $\mathsf{P}=P$. One can also readily verify in this case that $\mathsf{Q}=Q$ and $\mathsf{k}_{M,0}=\kappa_{M,0}$, so the matrices (2.1.2) and (2.2.3) are equal: $\mathsf{R}(u)=R(u)$. Hence, the assignment $\mathsf{t}(u)\mapsto T(u)$ yields an algebra isomorphism $\mathrm{X}(\mathfrak{so}_M)\xrightarrow{\sim}\mathrm{X}(\mathfrak{osp}_{M|0})$. Alternatively, when $M=0$ we have $\mathsf{P}=-P$, $\mathsf{Q}=-Q$, and $\mathsf{k}_{0,N}=-\kappa_{0,N}$; hence, $\mathsf{R}(u)=R(-u)$. Exchanging $(u,v)\mapsto(-u,-v)$ in the $RTT$-relation (2.2.7) therefore shows that $\mathsf{t}(u)\mapsto T(-u)$ induces an algebra isomorphism $\mathrm{X}(\mathfrak{sp}_N)\xrightarrow{\sim}\mathrm{X}(\mathfrak{osp}_{0|N})$.

In practice, to prove that some graded map $\varphi$ from $\mathrm{X}(\mathfrak{osp}_{M|N})$ to some superalgebra $\mathcal{A}$ is a superalgebra morphism, one sets $A_{ij}^{(n)}=\varphi(T_{ij}^{(n)})$ and collects these images into a matrix $A(u) = \sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij} \otimes A_{ij}(u) \in \mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathcal{A}[\![u^{-1}]\!]$, where $A_{ij}(u)$ is the series $\delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} A_{ij}^{(n)}u^{-n}$; thus, the map $\varphi\colon T(u) \mapsto A(u)$ describes the assignment $\varphi\colon T_{ij}^{(n)} \mapsto A_{ij}^{(n)}$. To show that the defining relations of the extended Yangian are satisfied when we replace the elements $T_{ij}^{(n)}$ with $A_{ij}^{(n)}$, one observes that such relations will be satisfied if and only if

$$R(u-v)A_1(u)A_2(v) = A_2(v)A_1(u)R(u-v).$$

Showing that $A(u)$ satisfies this latter form will be how we prove many graded maps are superalgebra morphisms from the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$.

As such, for any formal series $f = f(u) = 1 + \sum_{n=1}^{\infty} f_n u^{-n} \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ and any $a \in \mathbb{C}$, two important superalgebra automorphisms of $\mathrm{X}(\mathfrak{osp}_{M|N})$ are provided by the assignments

$$\mu_f\colon T(u) \mapsto f(u)T(u), \tag{2.2.9}$$

$$\tau_a\colon T(u) \mapsto T(u-a), \tag{2.2.10}$$

where one can show the above maps take the more explicit forms

$$\mu_f\colon T_{ij}^{(n)} \mapsto \sum_{a+b=n} f_a T_{ij}^{(b)} \quad \text{and} \quad \tau_a\colon T_{ij}^{(n)} \mapsto \sum_{k=1}^{n} \binom{n-1}{n-k} a^{n-k} T_{ij}^{(k)} \quad \text{for} \quad n \in \mathbb{Z}^+.$$

There also exists several important anti-automorphisms for the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$. To introduce such morphisms, we first note that we may regard $T(u)$ as a formal power series in $u^{-1}$ whose coefficients lie in $\mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathrm{X}(\mathfrak{osp}_{M|N})$. Since the constant term of such power series is the unit object $\mathbb{1} = \mathrm{id} \otimes \mathbf{1}$, then $T(u)$ must have an inverse $T(u)^{-1}$. Further, we shall understand $T^t(u)$ as $((-)^t \otimes \mathrm{id})T(u)$ and $T^{st}(u)$ as $((-)^{st} \otimes \mathrm{id})T(u)$. Hence, by interpreting $T(u) = \big(T_{ij}(u)\big)_{i,j=1}^{M+N}$ as a matrix in $\mathrm{Mat}_{M+N}\big(\mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]\big)$, then

$$T^t(u) = \big((-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{j}\bar{\imath}}(u)\big)_{i,j=1}^{M+N} \quad \text{and} \quad T^{st}(u) = \big((-1)^{[i][j]+[i]}T_{ji}(u)\big)_{i,j=1}^{M+N},$$

so we accordingly define the following for all $1 \le i, j \le M+N$:

$$T_{ij}^t(u) := (-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{j}\bar{\imath}}(u) \quad \text{and} \quad T_{ij}^{st}(u) := (-1)^{[i][j]+[i]}T_{ji}(u). \tag{2.2.11}$$

In particular, the assignments

$$\varsigma\colon T(u) \mapsto T(-u), \tag{2.2.12}$$
$$t\colon T(u) \mapsto T^t(u), \tag{2.2.13}$$
$$st\colon T(u) \mapsto T^{st}(u), \tag{2.2.14}$$
$$S\colon T(u) \mapsto T(u)^{-1}, \tag{2.2.15}$$

define superalgebra anti-automorphisms of $\mathrm{X}(\mathfrak{osp}_{M|N})$, c.f. [Mol07, Proposition 1.3.3]. For instance, proving that a graded map $(-)^\circ\colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathrm{X}(\mathfrak{osp}_{M|N})$ is a superalgebra anti-morphism is equivalent to showing the relation

$$R(u-v)T_2^\circ(v)T_1^\circ(u) = T_1^\circ(u)T_2^\circ(v)R(u-v),$$

where $T^\circ(u) = \sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij} \otimes T_{ij}^\circ(u)$ and $T_k^\circ(u)$, $k=1,2$, are defined in the suitable ways. For the maps (2.2.12)–(2.2.15), one can obtain the above relation by modifying the $RTT$-relation (2.2.7) in suitable ways and using the unitarity property of the $R$-matrix $R(u-v)$ along with the fact that $R(u-v)$ is invariant under the operators $(-)^t \otimes (-)^t$ and $(-)^{st} \otimes (-)^{st}$.

### 2.2.2 The Hopf structure and central series $\mathscr{Z}(u)$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$

As was first stated in [AAC+03, §3] with proof similar to [Mol07, Theorem 1.5.1], the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ comes equipped with a Hopf superalgebra structure given by the comultiplication

$$\Delta\colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathrm{X}(\mathfrak{osp}_{M|N}) \otimes \mathrm{X}(\mathfrak{osp}_{M|N}), \quad T(u) \mapsto T_{[1]}(u)T_{[2]}(u),$$

the counit

$$\varepsilon\colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathbb{C}, \quad T(u) \mapsto \mathbb{1},$$

and the antipode

$$S\colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathrm{X}(\mathfrak{osp}_{M|N}), \quad T(u) \mapsto T(u)^{-1},$$

which was previously introduced as the anti-automorphism (2.2.15). On the level of power series, the comultiplication $\Delta$ and counit $\varepsilon$ take the form

$$\Delta\colon T_{ij}(u) \mapsto \sum_{k=1}^{M+N} T_{ik}(u) \otimes T_{kj}(u) \quad \text{and} \quad \varepsilon\colon T_{ij}(u) \mapsto \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq M+N.$$

Hence, these maps can be written explicitly as $\Delta(T_{ij}^{(n)}) = \sum_{k=1}^{M+N} \sum_{a+b=n} T_{ik}^{(a)} \otimes T_{kj}^{(b)}$ and $\varepsilon(T_{ij}^{(n)}) = 0$ for $1 \leq i, j \leq M+N$ and $n \in \mathbb{Z}^+$. To compute how the antipode $S$ maps such generators, one can write

$$T(u)^{-1} = \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes T_{ij}^{\bullet}(u),$$

where $T_{ij}^{\bullet}(u) = \mathbf{1} + \sum_{n=1}^{\infty} T_{ij}^{\bullet(n)} u^{-n}$ are uniquely determined series in $\mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$. In particular, $S(T_{ij}^{(n)}) = T_{ij}^{\bullet(n)}$ and one can verify such images are of the form

$$T_{ij}^{\bullet(n)} = -T_{ij}^{(n)} + \sum_{s=2}^{n} (-1)^s \sum_{\sum_{j=1}^{s} k_j = n} \left( \sum_{a_1, a_2, \ldots, a_{s-1}=1}^{M+N} T_{ia_1}^{(k_1)} T_{a_1 a_2}^{(k_2)} \cdots T_{a_{s-1} j}^{(k_s)} \right)$$

with $k_j \in \mathbb{Z}^+$ for each $k_j$ in the sum $\sum_{j=1}^{s} k_j = n$.

Let us define the matrix

$$Z(u) := T^t(u+\kappa) T(u) \tag{2.2.16}$$

and further consider the series $\mathcal{Z}(u)$ lying in $\mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ such that $\mathrm{id} \otimes \mathcal{Z}(u) = Z(u)$. Multiplying both sides of the $RTT$-relation by $u-v-\kappa$, setting $u = v+\kappa$ and replacing $v$ by $u$ yields the equation

$$Q T_1(u+\kappa) T_2(u) = T_2(u) T_1(u+\kappa) Q.$$

Using the relations $QT_1(u) = QT_2^t(u)$ and $T_1(u)Q = T_2^t(u)Q$, transposing the first tensor factor of the above equation yields

$$P \otimes \mathcal{Z}(u) = P T_2^t(u+\kappa) T_2(u) = T_2(u) T_2^t(u+\kappa) P.$$

Multiplying the above equation on left by $P$ gives $\mathrm{id}^{\otimes 2} \otimes \mathcal{Z}(u) = T_2^t(u+\kappa)T_2(u)$, whilst instead multiplying on the right by $P$ yields $\mathrm{id}^{\otimes 2} \otimes \mathcal{Z}(u) = T_2(u)T_2^t(u+\kappa)$. Thus,

$$Z(u) = T^t(u+\kappa)T(u) = T(u)T^t(u+\kappa), \tag{2.2.17}$$

or rather put,

$$\delta_{ij}\mathcal{Z}(u) = \sum_{k=1}^{M+N} T_{ik}^t(u+\kappa)T_{kj}(u) = \sum_{k=1}^{M+N} T_{ik}(u)T_{kj}^t(u+\kappa), \tag{2.2.18}$$

where we shall write $\mathcal{Z}(u) = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Z}_n u^{-n} \in \mathbf{1} + u^{-1}\mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$.

We note that the coefficients of $\mathcal{Z}(u)$ are homogeneous of even degree, so such coefficients lie within the even subalgebra of $\mathrm{X}(\mathfrak{osp}_{M|N})$. We shall let $\mathrm{ZX}(\mathfrak{osp}_{M|N})$ denote the subalgebra generated by the coefficients of $\mathcal{Z}(u)$ and let $(\mathcal{Z}(u) - \mathbf{1})$ to mean the two-sided graded ideal of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $\mathcal{Z}(u) - \mathbf{1}$.

**Proposition 2.2.4.** *The coefficients of the series* $\mathcal{Z}(u) \in \mathbf{1} + u^{-1}\mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ *given by the equation* $T^t(u+\kappa)T(u) = T(u)T^t(u+\kappa) = \mathrm{id} \otimes \mathcal{Z}(u)$ *lie in the center of* $\mathrm{X}(\mathfrak{osp}_{M|N})$. *Furthermore,*

$$\Delta\colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u) \otimes \mathcal{Z}(u) \tag{2.2.19}$$

*where* $\Delta$ *is the comultiplication map. In particular,* $\mathrm{ZX}(\mathfrak{osp}_{M|N})$ *is a sub-Hopf superalgebra and* $(\mathcal{Z}(u) - \mathbf{1})$ *is a graded Hopf ideal of* $\mathrm{X}(\mathfrak{osp}_{M|N})$.

*Proof.* The proof was provided in [AAC+03, Theorem 3.1], but we will reproduce the argument here. First, one observes

$$Z(u)T_2(v) = T_1^t(u+\kappa)T_1(u)T_2(v) = T_1^t(u+\kappa)R(u-v)^{-1}T_2(v)T_1(u)R(u-v)$$

using the $RTT$-relation. By transposing the first tensor factor of the $RTT$-relation (2.2.7) and using properties (2.2.5) and (2.2.6), we also get the equation

$$T_1^t(u+\kappa)R(u-v)^{-1}T_2(v) = T_2(v)R(u-v)^{-1}T_1^t(u+\kappa).$$

Therefore,

$$\begin{aligned} Z(u)T_2(v) &= T_2(v)R(u-v)^{-1}T_1^t(u+\kappa)T_1(u)R(u-v) \\ &= T_2(v)R(u-v)^{-1}Z(u)R(u-v) = T_2(v)Z(u), \end{aligned}$$

since $Z(u)$ commutes with $R(u-v)$. Furthermore, $\Delta\colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u) \otimes \mathcal{Z}(u)$ is readily verified from the computation

$$\begin{aligned}\Delta(\mathcal{Z}(u)) &= \sum_{a,b,k=1}^{M+N} (-1)^{[i][k]+[k]}\theta_i\theta_k \big(T_{\bar{k}a}(u+\kappa)\otimes T_{a\bar{i}}(u+\kappa)\big)\big(T_{kb}(u)\otimes T_{bi}(u)\big) \\ &= \sum_{a,b,k=1}^{M+N} (-1)^{([a]+[i])([a]+[b])} T^t_{\bar{a}k}(u+\kappa)T_{kb}(u)\otimes T^t_{i\bar{a}}(u+\kappa)T_{bi}(u) \\ &= \sum_{a,b=1}^{M+N} (-1)^{([a]+[i])([a]+[b])}\delta_{\bar{a}b}\mathcal{Z}(u)\otimes T^t_{i\bar{a}}(u+\kappa)T_{bi}(u) = \mathcal{Z}(u)\otimes\mathcal{Z}(u).\end{aligned}$$

Denoting $\mathcal{I} = (\mathcal{Z}(u) - \mathbf{1})$, one may verify that $\varepsilon\colon \mathcal{Z}(u)\mapsto 1$ and so $\varepsilon(\mathcal{I}) = 0$. Moreover, since $\Delta(\mathcal{Z}_n) = \sum_{a+b=n}\mathcal{Z}_a\otimes\mathcal{Z}_b$ (where $\mathcal{Z}_0 = \mathbf{1}$), then for any $X\in \mathrm{X}(\mathfrak{osp}_{M|N})$ one has $\Delta(X\mathcal{Z}_n), \Delta(\mathcal{Z}_n X)\in\mathcal{I}\otimes\mathrm{X}(\mathfrak{osp}_{M|N}) + \mathrm{X}(\mathfrak{osp}_{M|N})\otimes\mathcal{I}$, showing that $\mathcal{I}$ is a coideal. Lastly, the axioms of a Hopf superalgebra structure infer that the image of $\mathcal{Z}(u)$ under the antipode is given by

$$S\colon \mathcal{Z}(u)\mapsto\mathcal{Z}(u)^{-1},$$

which proves the proposition. □

By identifying $\mathcal{Z}(u)$ with $Z(u)$, equation (2.2.17) shows that the inverse of $T(u)$ is given by

$$T(u)^{-1} = \mathcal{Z}(u)^{-1}T^t(u+\kappa),$$

so the antipode $S$ is the mapping $T(u)\mapsto\mathcal{Z}(u)^{-1}T^t(u+\kappa)$. In particular, the square of the antipode is computed as

$$S^2\colon T(u)\mapsto\mathcal{Z}(u)\mathcal{Z}(u+\kappa)^{-1}T(u+2\kappa), \tag{2.2.20}$$

which will be relevant in the later subsection §2.4.1.

### 2.2.3 The associated graded superalgebra $\operatorname{gr}\mathrm{X}(\mathfrak{osp}_{M|N})$

We shall now consider two (ascending algebra) filtrations on $\mathrm{X}(\mathfrak{osp}_{M|N})$, denoted $\mathbf{E}\big(\mathrm{X}(\mathfrak{osp}_{M|N})\big) = \mathbf{E} = \{\mathbf{E}_n\}_{n\in\mathbb{N}}$ and $\mathbf{E}'\big(\mathrm{X}(\mathfrak{osp}_{M|N})\big) = \mathbf{E}' = \{\mathbf{E}'_n\}_{n\in\mathbb{N}}$, given via the respective filtration degree assignments

$$\deg_{\mathbf{E}} T^{(n)}_{ij} = n-1 \quad\text{and}\quad \deg_{\mathbf{E}'} T^{(n)}_{ij} = n \tag{2.2.21}$$

for all $1 \leq i, j \leq M+N$ and $n \in \mathbb{Z}^+$. From the defining relations (2.2.8), the associated graded superalgebra $\mathrm{gr}_{\mathbf{E}'} \mathrm{X}(\mathfrak{osp}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{E}'_n/\mathbf{E}'_{n-1}$ is supercommutative. Our attention will primarily focus on the first filtration $\mathbf{E}$ which will induce a more interesting associated graded superalgebra, denoted

$$\mathrm{gr}\, \mathrm{X}(\mathfrak{osp}_{M|N}) := \mathrm{gr}_{\mathbf{E}}\, \mathrm{X}(\mathfrak{osp}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{E}_n/\mathbf{E}_{n-1}.$$

We note that $\mathrm{gr}\, \mathrm{X}(\mathfrak{osp}_{M|N})$ inherits a $\mathbb{Z}_2$-graded structure from $\mathrm{X}(\mathfrak{osp}_{M|N})$ by assigning $\mathbb{Z}_2$-grade $[i]+[j]$ to the image $\overline{T}_{ij}^{(n)}$ of $T_{ij}^{(n)}$ in $\mathbf{E}_{n-1}/\mathbf{E}_{n-2}$. Furthermore, by endowing $\mathrm{X}(\mathfrak{osp}_{M|N})^{\otimes 2}$ with the tensor product filtration $\mathbf{E}^2 = \{\mathbf{E}^2_n\}_{n\in\mathbb{N}}$ and assigning $\mathbb{C}$ with the trivial filtration $\mathbf{C} = \{\mathbf{C}_n\}_{n\in\mathbb{N}}$, i.e.,

$$\mathbf{E}^2_n = \sum_{i+j=n} \mathbf{E}_i \otimes \mathbf{E}_j \quad \text{and} \quad \mathbf{C}_n = \mathbb{C} \qquad \text{for all} \quad n \in \mathbb{N},$$

one can verify that each of the Hopf superalgebra structure maps on $\mathrm{X}(\mathfrak{osp}_{M|N})$ will preserve their relative filtrations. In short, $\mathbf{E}$ is a Hopf filtration on $\mathrm{X}(\mathfrak{osp}_{M|N})$, so $\mathrm{gr}\, \mathrm{X}(\mathfrak{osp}_{M|N})$ is equipped with an $\mathbb{N}$-graded Hopf superalgebra structure given by the comultiplication

$$\begin{aligned} \mathrm{gr}\,\Delta\colon\ \mathrm{gr}\,\mathrm{X}(\mathfrak{osp}_{M|N}) &\to \mathrm{gr}\big(\mathrm{X}(\mathfrak{osp}_{M|N})^{\otimes 2}\big) \cong \big(\mathrm{gr}\,\mathrm{X}(\mathfrak{osp}_{M|N})\big)^{\otimes 2} \\ \overline{T}_{ij}^{(n)} &\mapsto \overline{T}_{ij}^{(n)} \otimes \mathbf{1} + \mathbf{1} \otimes \overline{T}_{ij}^{(n)}, \end{aligned}$$

the counit

$$\mathrm{gr}\,\varepsilon\colon\ \mathrm{gr}\,\mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathbb{C}, \quad \overline{T}_{ij}^{(n)} \mapsto 0,$$

and antipode

$$\mathrm{gr}\,S\colon\ \mathrm{gr}\,\mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathrm{gr}\,\mathrm{X}(\mathfrak{osp}_{M|N}), \quad \overline{T}_{ij}^{(n)} \mapsto \overline{T}_{ij}^{\bullet(n)} = -\overline{T}_{ij}^{(n)},$$

where $1 \leq i, j \leq M+N$ and $n \in \mathbb{Z}^+$.

Given a Lie superalgebra $\mathfrak{g}$, we recall that $\mathfrak{g}[z]$ denotes the *polynomial current Lie superalgebra* associated to $\mathfrak{g}$; that is, $\mathfrak{g}[z]$ is equal to $\mathfrak{g} \otimes \mathbb{C}[z]$ as a super vector space (where the indeterminate $z$ is of $\mathbb{Z}_2$-grade $\bar{0}$), and is equipped with the Lie superbracket

$$[X \otimes f(z), Y \otimes g(z)] := [X, Y] \otimes f(z)g(z) \quad \text{for } X, Y \in \mathfrak{g} \text{ and } f(z), g(z) \in \mathbb{C}[z].$$

Equivalently, $\mathfrak{g}[z]$ may be regarded as the Lie superalgebra of polynomial maps $f\colon \mathbb{C} \to \mathfrak{g}$ with Lie superbracket given point-wise. We note that $\mathfrak{g}[z]$ is an $\mathbb{N}$-graded Lie superalgebra $\bigoplus_{n\in\mathbb{N}} \mathfrak{g}[z]_n$, where $\mathfrak{g}[z]_n = \mathfrak{g}\otimes\mathbb{C}z^n$, and we shall use the identification $Xz^n = X\otimes z^n$ for elements in $\mathfrak{g}[z]$.

Given any Lie superalgebra $\mathfrak{a}$, we let $\mathfrak{U}(\mathfrak{a})$ denote its universal enveloping superalgebra; that is, $\mathfrak{U}(\mathfrak{a}) = T(\mathfrak{a})/I(\mathfrak{a})$, where $T(\mathfrak{a})$ is the tensor superalgebra of $\mathfrak{a}$ and $I(\mathfrak{a})$ is the two-sided ideal generated by elements of the form $X\otimes Y - (-1)^{[X][Y]}Y\otimes X - [X,Y]$ for homogeneous elements $X, Y \in \mathfrak{a}$. Furthermore, $\mathfrak{U}(\mathfrak{a})$ is endowed with a Hopf superalgebra structure given by structure maps

$$\begin{aligned} \Delta\colon \mathfrak{U}(\mathfrak{a}) &\to \mathfrak{U}(\mathfrak{a})\otimes\mathfrak{U}(\mathfrak{a}), & \varepsilon\colon \mathfrak{U}(\mathfrak{a}) &\to \mathbb{C}, & S\colon \mathfrak{U}(\mathfrak{a}) &\to \mathfrak{U}(\mathfrak{a}), \\ X &\mapsto X\otimes 1 + 1\otimes X & X &\mapsto 0 & X &\mapsto -X \end{aligned}$$

for all $X \in \mathfrak{a}$. In the case when $\mathfrak{a} = \mathfrak{g}[z]$ is a polynomial current Lie superalgebra, we see that $\mathfrak{U}(\mathfrak{g}[z])$ is an $\mathbb{N}$-graded superalgebra $\bigoplus_{n\in\mathbb{N}} \mathfrak{U}^n(\mathfrak{g}[z])$, where

$$\mathfrak{U}^n(\mathfrak{g}[z]) = \operatorname{span}_{\mathbb{C}}\big\{ \textstyle\prod_{a=1}^{\gamma} X_a z^{k_a} \mid \gamma\in\mathbb{Z}^+,\ X_a\in\mathfrak{g},\ \sum_{a=1}^{\gamma} k_a = n\big\}.$$

Consider a central extension $\mathfrak{osp}_{M|N}\oplus\mathfrak{z}_{\mathsf{c}}$ of $\mathfrak{osp}_{M|N}$ by a purely even 1-dimensional abelian Lie superalgebra $\mathfrak{z}_{\mathsf{c}} := \mathbb{C}\cdot\mathsf{c}$. As a Lie superalgebra, $\mathfrak{osp}_{M|N}[z]\oplus\mathfrak{z}_{\mathsf{c}}[z]$ is generated by the elements $\{F_{ij}z^m, \mathsf{c}z^n \mid 1\le i,j\le M+N,\ m,n\in\mathbb{N}\}$ subject only to the relations

$$\begin{aligned} [F_{ij}z^m, F_{kl}z^n] &= \delta_{jk}F_{il}z^{m+n} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}F_{kj}z^{m+n} \\ &\quad - \delta_{\bar{i}k}(-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}l}z^{m+n} + \delta_{\bar{j}l}(-1)^{([i]+[j])[k]}\theta_{\bar{i}}\theta_{\bar{j}}F_{k\bar{i}}z^{m+n}, \\ F_{ij}z^n + (-1)^{[i][j]+[i]}\theta_i\theta_j F_{\bar{j}\bar{i}}z^n &= 0, \qquad \text{and} \qquad \big[F_{ij}z^m, \mathsf{c}z^n\big] = 0. \end{aligned}$$

We will now aim to construct a Hopf superalgebra epimorphism from the universal enveloping superalgebra $\mathfrak{U}(\mathfrak{osp}_{M|N}[z]\oplus\mathfrak{z}_{\mathsf{c}}[z])$ to the associated graded superalgebra $\operatorname{gr}\mathrm{X}(\mathfrak{osp}_{M|N})$. Before doing so, we note the defining equation (2.2.18) for the central elements $\mathcal{Z}_n$ infers

$$\delta_{ij}\mathcal{Z}_n \equiv T_{ij}^{(n)} + (-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{j}\bar{i}}^{(n)} \mod \mathbf{E}_{n-2}. \tag{2.2.22}$$

In particular, $\mathcal{Z}_n$ has filtration degree $n-1$, so we shall let $\overline{\mathcal{Z}}_n$ denote the image of $\mathcal{Z}_n$ in the graded component $\mathbf{E}_{n-1}/\mathbf{E}_{n-2}$. We can now describe the desired map as in the following proposition.

**Proposition 2.2.5.** *The map* $\Psi\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]\oplus \mathfrak{z}_{\mathsf{c}}[z]) \to \operatorname{gr}\mathrm{X}(\mathfrak{osp}_{M|N})$ *defined by*

$$F_{ij}z^{n-1} \mapsto (-1)^{[i]}\big(\overline{T}_{ij}^{(n)} - \tfrac{1}{2}\delta_{ij}\overline{\mathcal{Z}}_n\big), \quad \mathsf{c}z^{n-1} \mapsto \tfrac{1}{2}\overline{\mathcal{Z}}_n$$

*for all* $1 \le i,j \le M+N$ *and* $n \in \mathbb{Z}^+$, *is an epimorphism of* $\mathbb{N}$*-graded Hopf superalgebras.*

*Proof.* To show $\Psi\colon \mathfrak{osp}_{M|N}[z]\oplus\mathfrak{z}_{\mathsf{c}}[z] \to \operatorname{Lie}(\operatorname{gr}\mathrm{X}(\mathfrak{osp}_{M|N}))$ is an $\mathbb{N}$-graded Lie superalgebra morphism, one passes the defining relations (2.2.8) to the associated graded superalgebra to yield the relations

$$\begin{aligned}\big[\overline{T}_{ij}^{(m)},\overline{T}_{kl}^{(n)}\big] &= \delta_{jk}(-1)^{[k]}\overline{T}_{il}^{(m+n-1)} - \delta_{il}(-1)^{([i]+[j])[k]+[j][l]}\overline{T}_{kj}^{(m+n-1)}\\ &\quad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]+[j]}\theta_i\theta_j\overline{T}_{\bar{\jmath}l}^{(m+n-1)} + \delta_{\bar{\jmath}l}(-1)^{([i]+[j])[k]+[j]}\theta_i\theta_j\overline{T}_{k\bar{\imath}}^{(m+n-1)}.\end{aligned}$$

for $1 \le i,j,k,l \le M+N$ and $m,n\in\mathbb{Z}^+$. Hence, the desired relations follow from multiplying the above equation by the scalar $(-1)^{[i]+[k]}$, using that the elements $\overline{\mathcal{Z}}_n$, $n\in\mathbb{Z}^+$, are central, and incorporating the equivalence (2.2.22).

Hence, $\Psi$ extends to a superalgebra morphism $\mathfrak{U}(\mathfrak{osp}_{M|N}[z]\oplus\mathfrak{z}_{\mathsf{c}}[z]) \to \operatorname{gr}\mathrm{X}(\mathfrak{osp}_{M|N})$, which is also $\mathbb{N}$-graded. Such morphism is surjective since $\operatorname{gr}\mathrm{X}(\mathfrak{osp}_{M|N})$ is generated by the elements $\overline{T}_{ij}^{(n)}$ and the morphism $\Psi$ sends $(-1)^{[i]}F_{ij}z^{n-1}$ to $\overline{T}_{ij}^{(n)}$ for $i\neq j$ and maps $((-1)^{[k]}F_{kk}+\mathsf{c})z^{n-1}$ to $\overline{T}_{kk}^{(n)}$.

Lastly, it can be seen that $\Psi$ is a morphism of Hopf superalgebras from the descriptions of those Hopf superstructures on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z]\oplus\mathfrak{z}_{\mathsf{c}}[z])$ and $\operatorname{gr}\mathrm{X}(\mathfrak{osp}_{M|N})$ as before. □

### 2.2.4 Orthosymplectic Yangians

We are now in position to define the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$:

**Definition 2.2.6.** The *Yangian* $\mathrm{Y}(\mathfrak{osp}_{M|N})$ of $\mathfrak{osp}_{M|N}$ is the quotient of $\mathrm{X}(\mathfrak{osp}_{M|N})$ by the graded ideal $(\mathcal{Z}(u)-\mathbf{1})$, i.e.,

$$\mathrm{Y}(\mathfrak{osp}_{M|N}) := \mathrm{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u)-\mathbf{1}). \tag{2.2.23}$$

Equivalently, $\mathrm{Y}(\mathfrak{osp}_{M|N})$ is the unital associative $\mathbb{C}$-superalgebra on the generators $\{\mathcal{T}_{ij}^{(n)} \mid 1 \leq i,j \leq M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $[\mathcal{T}_{ij}^{(n)}] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the *RTT-relation*

$$\begin{gathered} R(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)R(u-v) \\ \text{in} \quad \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{osp}_{M|N})[\![u^{\pm 1}, v^{\pm 1}]\!], \end{gathered} \tag{2.2.24}$$

where $R(u-v)$ is identified with $R(u-v)\otimes \mathbf{1}$, and

$$\mathcal{T}^t(u+\kappa)\mathcal{T}(u) = \mathbb{1} \quad \text{in} \quad \operatorname{End}(\mathbb{C}^{M|N}) \otimes \mathrm{Y}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!], \tag{2.2.25}$$

where $\mathcal{T}(u) := \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes \mathcal{T}_{ij}(u) \in \operatorname{End}(\mathbb{C}^{M|N}) \otimes \mathrm{Y}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ is the matrix consisting of the series $\mathcal{T}_{ij}(u) := \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} \mathcal{T}_{ij}^{(n)} u^{-n} \in \mathrm{Y}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$ for indices $1 \leq i,j \leq M+N$.

**Remark 2.2.7.** When $N=0$, the non-super and super $R$-matrices (2.1.2) and (2.2.3) coincide: $\mathsf{R}(u) = R(u)$. In particular, the assignment $\mathsf{t}(u) \mapsto \mathcal{T}(u)$ yields an algebra isomorphism $\mathrm{Y}(\mathfrak{so}_M) \xrightarrow{\sim} \mathrm{Y}(\mathfrak{osp}_{M|0})$. Alternatively, when $M=0$ then there is an equality $\mathsf{R}(u) = R(-u)$. Exchanging $(u,v) \mapsto (-u,-v)$ in the *RTT*-relation (2.2.24) therefore shows $\mathsf{t}(u) \mapsto \mathcal{T}(-u)$ induces an algebra isomorphism $\mathrm{Y}(\mathfrak{sp}_N) \xrightarrow{\sim} \mathrm{Y}(\mathfrak{osp}_{0|N})$.

The defining relations for the Yangian in terms of formal power series equivalently take the form

$$\begin{aligned} \left[\mathcal{T}_{ij}(u), \mathcal{T}_{kl}(v)\right] = {} & \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(\mathcal{T}_{kj}(u)\mathcal{T}_{il}(v) - \mathcal{T}_{kj}(v)\mathcal{T}_{il}(u)\Big) \\ & - \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{p=1}^{M+N}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p\mathcal{T}_{pj}(u)\mathcal{T}_{\bar{p}l}(v) \\ & \qquad - \delta_{\bar{\jmath}l}\sum_{p=1}^{M+N}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p\mathcal{T}_{k\bar{p}}(v)\mathcal{T}_{ip}(u)\Bigg), \end{aligned} \tag{2.2.26}$$

and

$$\sum_{p=1}^{M+N} \mathcal{T}_{ip}^t(u+\kappa)\mathcal{T}_{pj}(u) = \delta_{ij}\mathbf{1}, \tag{2.2.27}$$

for all indices $1 \leq i,j,k,l \leq M+N$.

Note that since $(\mathcal{Z}(u) - \mathbf{1})$ is a graded Hopf ideal, the quotient of $\mathrm{X}(\mathfrak{osp}_{M|N})$ by $(\mathcal{Z}(u) - \mathbf{1})$ comes equipped with a unique Hopf superstructure such that the canonical projection $\mathrm{X}(\mathfrak{osp}_{M|N}) \twoheadrightarrow \mathrm{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u) - \mathbf{1})$ is a morphism of Hopf superalgebras. Hence, there is Hopf superalgebra structure on $\mathrm{Y}(\mathfrak{osp}_{M|N})$ given by the comultiplication

$$\Delta\colon \mathrm{Y}(\mathfrak{osp}_{M|N}) \to \mathrm{Y}(\mathfrak{osp}_{M|N}) \otimes \mathrm{Y}(\mathfrak{osp}_{M|N}), \quad \mathcal{T}(u) \mapsto \mathcal{T}_{[1]}(u)\mathcal{T}_{[2]}(u),$$

the counit

$$\varepsilon\colon \mathrm{Y}(\mathfrak{osp}_{M|N}) \to \mathbb{C}, \quad \mathcal{T}(u) \mapsto \mathbb{1},$$

and the antipode

$$S\colon \mathrm{Y}(\mathfrak{osp}_{M|N}) \to \mathrm{Y}(\mathfrak{osp}_{M|N}), \quad \mathcal{T}(u) \mapsto \mathcal{T}(u)^{-1} = \mathcal{T}^t(u+\kappa).$$

Furthermore, the filtrations $\mathbf{E}$ and $\mathbf{E}'$ on $\mathrm{X}(\mathfrak{osp}_{M|N})$ will endow the respective filtrations $\overline{\mathbf{E}} = \{\overline{\mathbf{E}}_n\}_{n\in\mathbb{N}}$ and $\overline{\mathbf{E}}' = \{\overline{\mathbf{E}}'_n\}_{n\in\mathbb{N}}$ on the quotient $\mathrm{X}(\mathfrak{osp}_{M|N})/(\mathcal{Z}(u) - \mathbf{1})$ such that

$$\overline{\mathbf{E}}_n = \mathbf{E}_n/\big(\mathbf{E}_n \cap (\mathcal{Z}(u) - \mathbf{1})\big) \quad \text{and} \quad \overline{\mathbf{E}}'_n = \mathbf{E}'_n/\big(\mathbf{E}'_n \cap (\mathcal{Z}(u) - \mathbf{1})\big).$$

For simplicity, we shall set $\mathbf{F} = \{\mathbf{F}_n\}_{n\in\mathbb{N}} := \overline{\mathbf{E}}$ and $\mathbf{F}' = \{\mathbf{F}'_n\}_{n\in\mathbb{N}} := \overline{\mathbf{E}}'$. In particular, these filtrations are given by the respective filtration degree assignments

$$\deg_{\mathbf{F}} \mathcal{T}_{ij}^{(n)} = n-1 \quad \text{and} \quad \deg_{\mathbf{F}'} \mathcal{T}_{ij}^{(n)} = n \tag{2.2.28}$$

for all $1 \leq i, j \leq M+N$ and $n \in \mathbb{Z}^+$. From the defining relations (2.2.26), one can deduce that the associated graded superalgebra $\mathrm{gr}_{\mathbf{F}'} \mathrm{Y}(\mathfrak{osp}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{F}'_n/\mathbf{F}'_{n-1}$ is supercommutative. Similar to the case of the extended Yangian, we will direct our attention to the first filtration $\mathbf{F}$ which will induce a more interesting associated graded superalgebra:

$$\mathrm{gr}\, \mathrm{Y}(\mathfrak{osp}_{M|N}) := \mathrm{gr}_{\mathbf{F}}\, \mathrm{Y}(\mathfrak{osp}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{F}_n/\mathbf{F}_{n-1},$$

The associated graded superalgebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{osp}_{M|N})$ inherits a $\mathbb{Z}_2$-graded structure from $\mathrm{Y}(\mathfrak{osp}_{M|N})$ by assigning $\mathbb{Z}_2$-grade $[i]+[j]$ to the image $\overline{\mathcal{T}}_{ij}^{(n)}$ of $\mathcal{T}_{ij}^{(n)}$ in $\mathbf{F}_{n-1}/\mathbf{F}_{n-2}$.

Again, similar to the subsection §2.2.3, one can verify that $\mathbf{F}$ is a Hopf filtration; hence, $\mathrm{gr}\,\mathrm{Y}(\mathfrak{osp}_{M|N})$ comes equipped with an $\mathbb{N}$-graded Hopf superalgebra structure $\mathrm{gr}\,\Delta$, $\mathrm{gr}\,\varepsilon$, $\mathrm{gr}\,S$ analogous to the one on $\mathrm{gr}\,\mathrm{X}(\mathfrak{osp}_{M|N})$. We note, however, that the antipode on $\operatorname{gr} \mathrm{Y}(\mathfrak{osp}_{M|N})$ takes on the form

$$\operatorname{gr} S(\overline{\mathcal{T}}_{ij}^{(n)}) = -\overline{\mathcal{T}}_{ij}^{(n)} = (-1)^{[i][j]+[j]}\theta_i\theta_j\overline{\mathcal{T}}_{\bar{j}\bar{i}}^{(n)}$$

for all $1 \leq i,j \leq M+N$ and $n \in \mathbb{Z}^+$.

Recalling the polynomial current superalgebra $\mathfrak{osp}_{M|N}[z]$ whose defining relations were described in the previous subsection, we obtain the following analogue of Proposition 2.2.5 for the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$:

**Proposition 2.2.8.** *There is an* $\mathbb{N}$*-graded Hopf superalgebra epimorphism*

$$\Phi \colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \to \operatorname{gr} \mathrm{Y}(\mathfrak{osp}_{M|N}), \quad F_{ij}z^{n-1} \mapsto (-1)^{[i]}\overline{\mathcal{T}}_{ij}^{(n)}$$

*for all* $1 \leq i,j \leq M+N$ *and* $n \in \mathbb{Z}^+$.

*Proof.* To show $\Phi \colon \mathfrak{osp}_{M|N}[z] \to \mathrm{Lie}(\operatorname{gr} \mathrm{Y}(\mathfrak{osp}_{M|N}))$ is an $\mathbb{N}$-graded Lie superalgebra morphism, one passes the relations (2.2.26) and (2.2.27) to the associated graded superalgebra to yield the respective relations

$$\begin{aligned}\big[\overline{\mathcal{T}}_{ij}^{(m)}, \overline{\mathcal{T}}_{kl}^{(n)}\big] = \delta_{jk}(-1)^{[k]}\overline{\mathcal{T}}_{il}^{(m+n-1)} - \delta_{il}(-1)^{([i]+[j])[k]+[j][l]}\overline{\mathcal{T}}_{kj}^{(m+n-1)} \\ - \delta_{\bar{i}k}(-1)^{[i][j]+[i]+[j]}\theta_i\theta_j\overline{\mathcal{T}}_{\bar{j}l}^{(m+n-1)} + \delta_{\bar{j}l}(-1)^{([i]+[j])[k]+[j]}\theta_i\theta_j\overline{\mathcal{T}}_{k\bar{i}}^{(m+n-1)}\end{aligned}$$

and

$$\overline{\mathcal{T}}_{ij}^{(n)} + (-1)^{[i][j]+[j]}\theta_i\theta_j\overline{\mathcal{T}}_{\bar{j}\bar{i}}^{(n)} = 0$$

for all $1 \leq i,j,k,l \leq M+N$ and $m,n \in \mathbb{Z}^+$. Hence, the desired relations follow from multiplying the first equation above by $(-1)^{[i]+[k]}$ and the second by $(-1)^{[i]}$.

Thus, $\Phi$ extends to a morphism of superalgebras $\mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \to \operatorname{gr} \mathrm{Y}(\mathfrak{osp}_{M|N})$ which is also $\mathbb{N}$-graded. This morphism is surjective since $\operatorname{gr} \mathrm{Y}(\mathfrak{osp}_{M|N})$ is generated by the elements $\overline{\mathcal{T}}_{ij}^{(n)}$.

Moreover, it can be seen that $\Phi$ is a morphism of Hopf superalgebras from the descriptions of the Hopf superstructures on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ and $\operatorname{gr} \mathrm{Y}(\mathfrak{osp}_{M|N})$. □

## 2.3 Poincaré-Birkhoff-Witt Theorem for the Yangian

In this section, we illustrate how to obtain an explicit algebraic basis for the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ which amounts to proving that the Yangian is a filtered deformation of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$. Indeed, if such an isomorphism $\Phi\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \xrightarrow{\sim} \operatorname{gr} \mathrm{Y}(\mathfrak{osp}_{M|N})$ exists, then the Poincaré-Birkhoff-Witt Theorem for Lie superalgebras infers that one can construct a basis $\mathbf{B}$ for $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$, so any lift of $\Phi(\mathbf{B})$ will yield the desired basis for the Yangian.

### 2.3.1 Evaluation and vector representations

Consider the vector representation of $\mathfrak{U}(\mathfrak{osp}_{M|N})$ on the super vector space $\mathbb{C}^{M|N}$ as given by

$$\rho\colon \mathfrak{U}(\mathfrak{osp}_{M|N}) \to \operatorname{End} \mathbb{C}^{M|N}, \quad F_{ij} \mapsto E_{ij} - (-1)^{[i][j]+[i]}\theta_i\theta_j E_{\bar{j}\bar{i}}. \tag{2.3.1}$$

for all $1 \le i, j \le M+N$. Given any $a \in \mathbb{C}$, one can pullback the vector representation by the superalgebra morphism $\mathrm{ev}_a\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \to \mathfrak{U}(\mathfrak{osp}_{M|N})$ induced by the assignment $z \mapsto a$ to yield the *evaluation representation* of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ at $a \in \mathbb{C}$ given by

$$\rho_a := \mathrm{ev}_a^* \rho\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \to \operatorname{End} \mathbb{C}^{M|N}, \quad F_{ij}z^n \mapsto a^n \rho(F_{ij}).$$

For any complex numbers $a_1, \dots, a_n \in \mathbb{C}$, we may therefore consider the tensor product of such evaluation representations of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$:

$$\rho_{a_1 \to a_n} := (\textstyle\bigotimes_{i=1}^n \rho_{a_i}) \circ \Delta_{n-1}, \tag{2.3.2}$$

where $\Delta_{n-1}\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \to \mathfrak{U}(\mathfrak{osp}_{M|N}[z])^{\otimes n}$ is the unique $(n-1)$-fold comultiplication sending $X \in \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ to the element $\sum_{(X)} X_{(1)} \otimes X_{(2)} \otimes \cdots \otimes X_{(n)}$ in Sweedler notation.

The following lemma establishes that the intersection of all kernels of such representations $\rho_{a_1 \to a_n}$, for all $a_1, \dots, a_n \in \mathbb{C}$, $n \in \mathbb{Z}^+$, is trivial. The core ideas for the proof arise from the proofs of analogous statements in the papers [Naz99, Proposition 2.2] and [AMR06, Lemma 3.5].

**Lemma 2.3.1.** $\bigcap_{n\in\mathbb{Z}^+}\bigcap_{(a_1,\dots,a_n)\in\mathbb{C}^n}\ker(\rho_{a_1\to a_n})=0$ *in* $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$.

*Proof.* Let $\{X_i\}_{i=1}^d$ be a homogeneous basis of $\mathfrak{osp}_{M|N}$ and write $\chi_i=\rho(X_i)$ for indices $i=1,2,\dots,d$. Furthermore, we shall let $\{\mathfrak{U}_n(\mathfrak{osp}_{M|N}[z])\}_{n\in\mathbb{N}}$ denote the canonical ascending algebra filtration on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ determined by monomial length.

*Step 1.* We start by endowing a total ordering '$\preceq$' on the collection of basis elements $\{X_bz^m \mid 1\le b\le d,\ m\in\mathbb{N}\}$ of $\mathfrak{osp}_{M|N}[z]$, so via the Poincaré-Birkhoff-Witt Theorem for Lie superalgebras, the universal enveloping superalgebra $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ has a basis consisting of ordered monomials of the form $\prod_{j=1}^r X_{b_j}z^{m_j}$ such that $X_{b_j}z^{m_j}\preceq X_{b_{j+1}}z^{m_{j+1}}$ for indices $j=1,\dots,r-1$, and $X_{b_j}z^{m_j}\neq X_{b_{j+1}}z^{m_{j+1}}$ provided $[X_{b_j}]=\bar{1}$. Given a nonzero element $A$ in $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$, we can therefore express such element as a unique linear combination of PBW basis monomials in $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ and we denote $\{M_i=\prod_{j=1}^n X_{b_{ij}}z^{m_{ij}}\}_{i=1}^p$ as the collection of those basis elements with maximal filtration degree $n$. For every such maximal length monomial, we consider their corresponding supersymmetrized object

$$M_i^\sigma := \sum_{\sigma\in\mathfrak{S}_n}(-1)^{\epsilon(\sigma,M_i)}\bigotimes_{j=1}^n X_{b_{i\sigma(j)}}z^{m_{i\sigma(j)}}\in(\mathfrak{osp}_{M|N}[z])^{\otimes n}, \tag{2.3.3}$$

where $(-1)^{\epsilon(\sigma,M_i)}$ is the Koszul sign provided that $\mathfrak{S}_n$ is the symmetric group on $n$ letters and $\epsilon\colon \mathfrak{S}_n\times(\mathfrak{osp}_{M|N}[z])^{\otimes n}\to\mathbb{Z}_2$ is the map $\epsilon(\sigma,x)=\sum_{(k,l)\in\mathrm{Inv}(\sigma)}[x_{\sigma(k)}][x_{\sigma(l)}]$ on homogeneous tensors $x=x_1\otimes x_2\otimes\cdots\otimes x_n\in(\mathfrak{osp}_{M|N}[z])^{\otimes n}$, where $\mathrm{Inv}(\sigma)$ is the set of inversions $\{(k,l)\mid k<l,\ \sigma(k)>\sigma(l)\}$.

*Step 2.* We now show that the $p$ supersymmetrized elements (2.3.3) are linearly independent, which amounts to proving such is true for their images under the projection $T(\mathfrak{osp}_{M|N}[z])\twoheadrightarrow\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$. To start, we first express each monomial in the sum $\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\epsilon(\sigma,M_i)}\prod_{j=1}^n X_{b_{i\sigma(j)}}z^{m_{i\sigma(j)}}$ in terms of the PBW basis for $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ with respect to the total order '$\preceq$'. By repeated use of the defining relations of the universal enveloping algebra, one yields that

$$\prod_{j=1}^n X_{b_{i\sigma(j)}}z^{m_{i\sigma(j)}}=(-1)^{\epsilon(\sigma,M_i)}\prod_{j=1}^n X_{b_{ij}}z^{m_{ij}} \mod \mathfrak{U}_{n-1}(\mathfrak{osp}_{M|N}[z]).$$

Therefore, the linear independence of the supersymmetrized elements (2.3.3) amounts to whether or not there is a non-trivial solution to

$$\sum_{i=1}^{p} \lambda_i \prod_{j=1}^{n} X_{b_{ij}} z^{m_{ij}} \equiv 0 \mod \mathfrak{U}_{n-1}(\mathfrak{osp}_{M|N}[z]),$$

but this is not possible unless $\lambda_i = 0$ for all $i = 1, 2, \ldots, p$.

*Step 3.* Since

$$\rho_{a_1 \to a_n}(X_b z^m) = \sum_{k=1}^{n} a_k^m \chi_b^{[k]}, \qquad \chi_b^{[k]} := \mathrm{id}^{\otimes(k-1)} \otimes \chi_b \otimes \mathrm{id}^{\otimes(n-k)} \in \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n},$$

then the image of the any monomial $\prod_{j=1}^{r} X_{b_j} z^{m_j}$ under $\rho_{a_1 \to a_n}$ is given by

$$\sum_{k_1,\ldots,k_r=1}^{n} a_{k_1}^{m_1} \cdots a_{k_r}^{m_r} \chi_{b_1}^{[k_1]} \cdots \chi_{b_r}^{[k_r]} \in \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}. \tag{2.3.4}$$

By completing the collection $\{\chi_i\}_{i=1}^{d}$ to a homogeneous basis $\{\chi_i\}_{i=1}^{(M+N)^2}$ of $\mathrm{End}(\mathbb{C}^{M|N})$ such that $\chi_j = \mathrm{id}$ for some $d+1 \leq j \leq (M+N)^2$, we consider the subspace of $\mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}$ given by

$$W_n := \mathrm{span}_{\mathbb{C}} \left\{ \chi_{i_1} \otimes \cdots \otimes \chi_{i_n} \mid \chi_j = \mathrm{id} \text{ occurs in at least one tensor factor} \right\},$$

where $1 \leq i_k \leq (M+N)^2$ for $1 \leq k \leq n$. We observe that the image of any element in $\mathfrak{U}_{n-1}(\mathfrak{osp}_{M|N}[z])$ under $\rho_{a_1 \to a_n}$ will be contained in the subspace $W_n$. Moreover, by (2.3.4) the image of the monomial $M_i$ under $\rho_{a_1 \to a_n}$ may be written as

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma, M_i)} a_1^{m_{i\sigma(1)}} \cdots a_n^{m_{i\sigma(n)}} \bigotimes_{j=1}^{n} \chi_{b_{i\sigma(j)}} \mod W_n. \tag{2.3.5}$$

Under the identification $\phi\colon (\mathfrak{osp}_{M|N}[z])^{\otimes n} \xrightarrow{\sim} (\mathfrak{osp}_{M|N})^{\otimes n} [z_1, \ldots, z_n]$, the images of the supersymmetrized elements $M_i^{\sigma}$ under $\phi$ are given by

$$\sum_{\sigma \in \mathfrak{S}_n} \big( (-1)^{\epsilon(\sigma, M_i)} \bigotimes_{j=1}^{n} X_{b_{i\sigma(j)}} \big) z_1^{m_{i\sigma(1)}} \cdots z_n^{m_{i\sigma(n)}}. \tag{2.3.6}$$

Since $\rho$ is a faithful representation, then so is $\rho^{\otimes n}\colon \mathfrak{U}(\mathfrak{osp}_{M|N})^{\otimes n} \to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}$ and its extension to $\mathfrak{U}(\mathfrak{osp}_{M|N})^{\otimes n}[z_1, \ldots, z_n] \to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}[z_1, \ldots, z_n]$, which we also denote $\rho^{\otimes n}$. Furthermore, since the elements $\phi(M_i^\sigma)$, $i = 1, \dots, p$, are linearly independent, then their images under $\rho^{\otimes n}$ are so. Hence, a nonzero linear combination $\sum_{i=1}^p \lambda_i \phi(M_i^\sigma)$ implies that the sum of polynomials

$$\sum_{i=1}^p \lambda_i \sum_{\sigma \in \mathfrak{S}_n} \Big( (-1)^{\epsilon(\sigma, M_i)} \bigotimes_{j=1}^n \chi_{b_{i\sigma(j)}} \Big) z_1^{m_{i\sigma(1)}} \cdots z_n^{m_{i\sigma(n)}}$$

is nonzero. Thus, there exists complex numbers $a_1, \dots, a_n \in \mathbb{C}$ such that

$$\sum_{i=1}^p \lambda_i \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma, M_i)} a_1^{m_{i\sigma(1)}} \cdots a_n^{m_{i\sigma(n)}} \bigotimes_{j=1}^n \chi_{b_{i\sigma(j)}}$$

is nonzero. Comparing the above with (2.3.5), we conclude that that image of $\rho_{a_1 \to a_n}(A)$ in the quotient $\mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}/W_n$ is nonzero and therefore $\rho_{a_1 \to a_n}(A) \neq 0$, proving the lemma. □

We will now direct our attention to an important representation of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ called the *vector representation.* The vector representation will play an important role in study of the representation theory of $\mathrm{X}(\mathfrak{osp}_{M|N})$ in the next chapter; however, it is relevant for this section since it will give rise to a vector representation of the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ which will be used to prove the isomorphism $\mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \cong \mathrm{gr}\, \mathrm{Y}(\mathfrak{osp}_{M|N})$.

The vector representation is built from a canonical representation of the extended Yangian which we will show here. By substituting $u \mapsto u - v$ in the SQYBE (2.2.4), one can readily verify that the assignment

$$R\colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathrm{End}\, \mathbb{C}^{M|N}, \quad T(u) \mapsto R(u) \tag{2.3.7}$$

defines a representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$, which we call the *$R$-matrix representation* of the extended Yangian. On the level of power series, the $R$-matrix representation takes the form

$$R\colon T_{ij}(u) \mapsto \delta_{ij}\, \mathrm{id} - \frac{(-1)^{[i][j]} E_{ji}}{u} + \frac{(-1)^{[j]} \theta_i \theta_j E_{\bar{\imath}\bar{\jmath}}}{u - \kappa}.$$

for all $1 \leq i, j \leq M+N$. A variant of the $R$-matrix representation is achieved by first twisting the action via the automorphism $st \circ \varsigma$ described by (2.2.12) and (2.2.14), yielding the representation $\varrho := R \circ st \circ \varsigma$ given on the level of power series by

$$\varrho(T_{ij}(u)) = (-1)^{[i][j]+[i]} R(T_{ji}(-u)) = \delta_{ij}\,\mathrm{id} + \frac{(-1)^{[i]} E_{ij}}{u} - \frac{(-1)^{[i][j]} \theta_i \theta_j E_{\bar{j}\bar{\imath}}}{u+\kappa}$$

for all $1 \le i, j \le M+N$, or in matrix form

$$\varrho\colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \operatorname{End} \mathbb{C}^{M|N}, \quad T(u) \mapsto R^{st}(-u), \tag{2.3.8}$$

where we identify $R^{st}(u)$ with $R^{st_1}(u)$. We call $\varrho$ the *vector representation of* $\mathrm{X}(\mathfrak{osp}_{M|N})$. The pullback of $\varrho$ by the automorphism $\tau_a$ as in (2.2.10) will result in a representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ given by

$$\varrho_a := \tau_a^* \varrho\colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \operatorname{End} \mathbb{C}^{M|N}, \quad T(u) \mapsto R^{st}(a-u) \tag{2.3.9}$$

for any $a \in \mathbb{C}$. On the level of power series, such representation takes the form

$$\varrho_a\colon T_{ij}(u) \mapsto \delta_{ij}\,\mathrm{id} + \frac{(-1)^{[i]} E_{ij}}{u-a} - \frac{(-1)^{[i][j]} \theta_i \theta_j E_{\bar{j}\bar{\imath}}}{u+\kappa-a},$$

and we call $\varrho_a$ the *vector representation of* $\mathrm{X}(\mathfrak{osp}_{M|N})$ *at level* $a \in \mathbb{C}$. We will see that by composing the vector representation $\varrho_a$ with a suitable automorphism of $\mathrm{X}(\mathfrak{osp}_{M|N})$, the resulting representation will descend to one for $\mathrm{Y}(\mathfrak{osp}_{M|N})$, thereby giving an analogue of the vector representations for the Yangian.

**Proposition 2.3.2.** *If $\mathcal{A}$ is a commutative unital associative $\mathbb{C}$-algebra, then for any formal series $a(u) = \mathbf{1} + \sum_{n=1}^{\infty} a_n u^{-n} \in \mathbf{1} + u^{-1}\mathcal{A}[\![u^{-1}]\!]$ and any $k \in \mathbb{C}$, there exists a unique formal series $y(u) = \mathbf{1} + \sum_{n=1}^{\infty} y_n u^{-n} \in \mathbf{1} + u^{-1}\mathcal{A}[\![u^{-1}]\!]$ such that*

$$a(u) = y(u)y(u+k). \tag{2.3.10}$$

*Proof.* The argument is the same as in [MNO96, §2.15], [AMR06, Theorem 3.1]. By writing the equality (2.3.10) in terms of the coefficients of $a(u)$, we yield the relations

$$a_n = 2y_n + B_n(y_1, \dots, y_{n-1}) \qquad \text{for} \quad n \in \mathbb{Z}^+, \tag{2.3.11}$$

where $B_n$ is a quadratic polynomial in $n-1$ indeterminates over $\mathbb{C}$. One may then inductively solve for the coefficients of $y(u)$ since the above relation implies that $y_n$

will be a quadratic polynomial in $a_1, \dots, a_n$. By construction, such a series $y(u)$ is unique. □

Given $a \in \mathbb{C}$, Proposition 2.3.2 infers there exists a unique series $f_a(u)$ lying in $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that

$$f_a(u)f_a(u+\kappa) = \frac{(u+\kappa-a)^2}{(u+\kappa-a)^2-1}. \tag{2.3.12}$$

The pullback of the vector representation $\varrho_a$ (2.3.9) at level $a \in \mathbb{C}$ by the shift automorphism $\mu_{f_a}$ (2.2.10) yields a new representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ given by

$$\phi_a := \mu_{f_a}^* \varrho_a \colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \operatorname{End} \mathbb{C}^{M|N}, \quad T(u) \mapsto f_a(u) R^{st}(a-u),$$

where we are identifying $(-)^{st}$ with $(-)^{st_1}$. Using equation (2.2.17) and the fact that the super-transposition $(-)^t$ and super-transpose $(-)^{st}$ commute, we find $\phi_a(Z(u))$ is given by $f_a(u)f_a(u+\kappa)R^{st}(a-u)\big(R^t(a-u-\kappa)\big)^{st}$, where we similarly identify $(-)^t$ with $(-)^{t_1}$. Using the relations

$$(P^{st})^2 = (M-N)P^{st}, \quad P^{st}Q^{st} = Q^{st}P^{st} = P^{st}, \quad \text{and} \quad (Q^{st})^2 = \mathrm{id}^{\otimes 2},$$

we find $R^{st}(a-u)\big(R^t(a-u-\kappa)\big)^{st} = \frac{(u+\kappa-a)^2-1}{(u+\kappa-a)^2}\,\mathrm{id}^{\otimes 2}$; hence, $\phi_a(\mathcal{Z}(u)) = \mathrm{id}$, where $\mathcal{Z}(u)$ is the series defined by (2.2.18), and so $\phi_a$ descends to the representation

$$\varphi_a \colon \mathrm{Y}(\mathfrak{osp}_{M|N}) \to \operatorname{End} \mathbb{C}^{M|N}, \quad \mathcal{T}(u) \mapsto f_a(u) R^{st}(a-u), \tag{2.3.13}$$

called the *vector representation of* $\mathrm{Y}(\mathfrak{osp}_{M|N})$ *at level* $a \in \mathbb{C}$. When $a = 0$, we set $\varphi_0 = \varphi$ and simply refer to it as the *vector representation of* $\mathrm{Y}(\mathfrak{osp}_{M|N})$.

### 2.3.2 The PBW Theorem and supercenter of $\mathrm{Y}(\mathfrak{osp}_{M|N})$

We are now in pole position to prove the main theorem of Chapter 2; namely, that the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ is a filtered deformation of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$. The proof of the following theorem is similar to [AMR06, Theorem 3.6] and leverages the lemma proven in the previous subsection.

**Theorem 2.3.3.** *The epimorphism in Proposition 2.2.8 is an* $\mathbb{N}$*-graded Hopf superalgebra isomorphism*

$$\Phi\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \xrightarrow{\sim} \operatorname{gr} \mathrm{Y}(\mathfrak{osp}_{M|N}), \quad F_{ij}z^{n-1} \mapsto (-1)^{[i]}\overline{\mathcal{T}}_{ij}^{(n)} \tag{2.3.14}$$

*for all* $1 \leq i, j \leq M+N$ *and* $n \in \mathbb{Z}^{+}$.

*Proof.* By Proposition 2.2.8, all that is left to show is injectivity. To this end, we let $A \in \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ be a nonzero homogeneous element of gradation degree $d$; that is,

$$A = \sum A_{i_1j_1;\dots;i_mj_m}^{k_1;\dots;k_m} F_{i_1j_1}z^{k_1-1}\cdots F_{i_mj_m}z^{k_m-1} \quad \text{where} \quad A_{i_1j_1;\dots;i_mj_m}^{k_1;\dots;k_m} \in \mathbb{C},$$

and the summation indices $i_b, j_b, k_b$, $1 \leq b \leq m$, satisfy $1 \leq i_b, j_b \leq M+N$ and $\sum_{b=1}^{m} k_b = d+m$. Considering the element

$$\widetilde{A} = \sum (-1)^{\sum_{b=1}^{m}[i_b]} A_{i_1j_1;\dots;i_mj_m}^{k_1;\dots;k_m} \mathcal{T}_{i_1j_1}^{(k_1)}\cdots \mathcal{T}_{i_mj_m}^{(k_m)} \in \mathrm{Y}(\mathfrak{osp}_{M|N})$$

whose summation indices $i_b, j_b, k_b$, $1 \leq b \leq m$, satisfy the same conditions as above, then $\Phi(A)$ coincides with the image of $\widetilde{A}$ in $\operatorname{gr}\mathrm{Y}(\mathfrak{osp}_{M|N})$, so it suffices to prove that the filtration degree of $\widetilde{A}$ is $d$.

*Step 1.* Writing the series $f_a(u)$ in (2.3.12) as the sum $\sum_{n=0}^{\infty} f_a^{(n)}u^{-n}$, $f_a^{(0)} = 1$, the coefficient of $u^{-n}$ in $f_a(u)f_a(u+\kappa)$ is given by $2f_a^{(n)} + \sum_{p=1}^{n-1}\sum_{k=1}^{p}\binom{p-1}{p-k}(-\kappa)^{p-k}f_a^{(n-p)}f_a^{(p)}$. Furthermore, using the expansion $\frac{(u+\kappa-a)^2}{(u+\kappa-a)^2-1} = \sum_{p=0}^{\infty}(u+\kappa-a)^{-2p}$, where

$$\frac{1}{(u+\kappa-a)^{2p}} = u^{-2p}\left(\sum_{n=0}^{\infty}(a-\kappa)^n u^{-n}\right)^{2p} = \sum_{n=2p}^{\infty}\binom{n-1}{n-2p}(a-\kappa)^{n-2p}u^{-n},$$

we see that the coefficient of $u^{-n}$ in $\frac{(u+\kappa-a)^2}{(u+\kappa-a)^2-1}$ is given by $\sum_{p=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-1}{n-2p}(a-\kappa)^{n-2p}$. Regarding $a$ as a formal variable in $\mathbb{C}$, the defining relation for $f_a(u)$ infers that its coefficients are elements in $\mathbb{C}[a]$. In particular, the polynomial degree of $f_a^{(n)}$ is given by $\deg_a f_a^{(1)} = 0$ and $\deg_a f_a^{(n)} = n-2$ for $n \geq 2$.

Via the expansions $\frac{1}{u-a} = \sum_{n=0}^{\infty} a^n u^{-n-1}$ and $\frac{1}{u+\kappa-a} = \sum_{n=0}^{\infty}(a-\kappa)^n u^{-n-1}$, the image of $\mathcal{T}_{ij}^{(n)}$ under the representation (2.3.13) is given by

$$\varphi_a(\mathcal{T}_{ij}^{(n)}) = \delta_{ij}f_a^{(n)}\,\mathrm{id} + \sum_{r+s=n} f_a^{(r)}(-1)^{[i]}E_{ij}a^{s-1} - \sum_{r+s=n} f_a^{(r)}(-1)^{[i][j]}\theta_i\theta_j E_{\bar{j}\bar{\imath}}(a-\kappa)^{s-1},$$

where $r \in \mathbb{N}$ and $s, n \in \mathbb{Z}^+$. In particular, $\varphi_a(\mathcal{T}_{ij}^{(n)}) \in \mathrm{End}(\mathbb{C}^{M|N})[a]$ with polynomial degree $n-1$ where the coefficient of $a^{n-1}$ is precisely $(-1)^{[i]}\rho(F_{ij})$ as given by (2.3.1).

*Step 2.* Given complex numbers $x_1, \ldots, x_n \in \mathbb{C}$, we consider the tensor product $\varphi_{x_1 \to x_n} := (\bigotimes_{i=1}^n \varphi_{x_i}) \circ \Delta_{n-1}$. Equipping $\mathrm{Y}(\mathfrak{osp}_{M|N})^{\otimes n}$ with the tensor product filtration $\mathbf{F}^n = \{\mathbf{F}_h^n\}_{h \in \mathbb{N}}$ induced by the one on $\mathrm{Y}(\mathfrak{osp}_{M|N})$, i.e., $\mathbf{F}_h^n = \bigoplus_{\sum_{i=1}^n k_i = h} \mathbf{F}_{k_1} \otimes \cdots \otimes \mathbf{F}_{k_n}$, then writing the sum $\sum_{b=1}^m k_b = d + m$ allows one to express $\Delta_{n-1}(\mathcal{T}_{i_1 j_1}^{(k_1)} \cdots \mathcal{T}_{i_m j_m}^{(k_m)})$ as

$$\sum_{q_1, \ldots, q_m = 1}^{n} (\mathcal{T}_{i_1 j_1}^{(k_1)})_{[q_1]} \cdots (\mathcal{T}_{i_m j_m}^{(k_m)})_{[q_m]} \quad \mathrm{mod}\ \mathbf{F}_{d-1}^n,$$

where $(\mathcal{T}_{i_b j_b}^{(k_b)})_{[q_b]} = \mathbf{1}^{\otimes (q_b - 1)} \otimes \mathcal{T}_{i_b j_b}^{(k_b)} \otimes \mathbf{1}^{\otimes (n - q_b)}$ for $1 \le b \le m$.

Regarding $x_1, \ldots, x_n$ as formal variables taking values in $\mathbb{C}$, the image of the monomial $\mathcal{T}_{i_1 j_1}^{(k_1)} \cdots \mathcal{T}_{i_m j_m}^{(k_m)}$ under the representation $\varphi_{x_1 \to x_n}$ will lie in $\mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}[x_1, \ldots, x_n]$ with polynomial degree satisfying $\deg\big(\varphi_{x_1 \to x_n}(\mathcal{T}_{i_1 j_1}^{(k_1)} \cdots \mathcal{T}_{i_m j_m}^{(k_m)})\big) \le d$.

If $\mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}[x_1, \ldots, x_n]_{d-1}$ denotes the subspace of polynomials in $x_1, \ldots, x_n$ with degree at most $d-1$, the element $\varphi_{x_1 \to x_n}(\mathcal{T}_{i_1 j_1}^{(k_1)} \cdots \mathcal{T}_{i_m j_m}^{(k_m)})$ is equivalent modulo $\mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}[x_1, \ldots, x_n]_{d-1}$ to the expression

$$\sum_{q_1, \ldots, q_m = 1}^{n} (-1)^{\sum_{b=1}^m [i_b]} \rho(F_{i_1 j_1})_{[q_1]} \cdots \rho(F_{i_m j_m})_{[q_m]}\, x_{q_1}^{k_1 - 1} \cdots x_{q_m}^{k_m - 1}$$

where $\rho(F_{i_b j_b})_{[q_b]} = \mathrm{id}^{\otimes (q_b - 1)} \otimes \rho(F_{i_b j_b}) \otimes \mathrm{id}^{\otimes (n - q_b)}$ for $1 \le b \le m$. In particular, we have

$$\varphi_{x_1 \to x_n}(\widetilde{A}) \equiv \rho_{x_1 \to x_n}(A) \quad \mathrm{mod}\ \mathrm{End}(\mathbb{C}^{M|N})^{\otimes n}[x_1, \ldots, x_n]_{d-1},$$

where $\rho_{x_1 \to x_n}$ is the representation of $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ given by (2.3.2). By Lemma 2.3.1, there exists $a_1, \ldots, a_n \in \mathbb{C}$ such that $\rho_{a_1 \to a_n}(A) \neq 0$; thus, $\varphi_{x_1 \to x_n}(\widetilde{A})$ has polynomial degree $d$, so $\widetilde{A}$ is of filtration degree $d$. □

Given as Corollary 2.3.4, the explicit form of the Poincaré-Birkhoff-Witt-type theorem for the Yangian is an immediate consequence of Theorem 2.3.3 and the Poincaré-Birkhoff-Witt Theorem for Lie superalgebras.

**Corollary 2.3.4** (PBW Theorem for $\mathrm{Y}(\mathfrak{osp}_{M|N})$). *Let $\mathcal{B}_{M|N}$ be an index set of pairs $(i,j) \in (\mathbb{Z}^+)^2$ such that $\{F_{ij} \mid (i,j) \in \mathcal{B}_{M|N}\}$ forms a basis for $\mathfrak{osp}_{M|N}$. Given any total ordering '$\preceq$' on the set $\mathsf{Y} = \{\mathcal{T}_{ij}^{(n)} \mid (i,j,n) \in \mathcal{B}_{M|N} \times \mathbb{Z}^+\}$, the collection of all ordered monomials of the form*

$$\mathcal{T}_{i_1j_1}^{(n_1)}\mathcal{T}_{i_2j_2}^{(n_2)}\cdots\mathcal{T}_{i_kj_k}^{(n_k)},$$

*where $\mathcal{T}_{i_aj_a}^{(n_a)} \in \mathsf{Y}$, $\mathcal{T}_{i_aj_a}^{(n_a)} \preceq \mathcal{T}_{i_{a+1}j_{a+1}}^{(n_{a+1})}$, and $\mathcal{T}_{i_aj_a}^{(n_a)} \neq \mathcal{T}_{i_{a+1}j_{a+1}}^{(n_{a+1})}$ if $\mathcal{T}_{i_aj_a}^{(n_a)}$ is odd, constitutes a basis for the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$.*

To construct the index set $\mathcal{B}_{M|N}$, one may first find bases for $\mathfrak{so}_M \hookrightarrow \mathfrak{osp}_{M|N}$ and $\mathfrak{sp}_N \hookrightarrow \mathfrak{osp}_{M|N}$ and complement such with a basis for the odd subspace of $\mathfrak{osp}_{M|N}$. For instance, by setting

$$\mathcal{B}^M = \{(i,j) \in (\mathbb{Z}^+)^2 \mid 2 \leq i+j \leq M\}$$
$$\text{and} \quad \mathcal{B}_N = \{(i,j) \in (\mathbb{Z}^+)^2 \mid 2M+2 \leq i+j \leq 2M+N+1\},$$

the collections $\{F_{ij} \mid (i,j) \in \mathcal{B}^M\}$ and $\{F_{ij} \mid (i,j) \in \mathcal{B}_N\}$ form respective bases for $\mathfrak{so}_M \hookrightarrow \mathfrak{osp}_{M|N}$ and $\mathfrak{sp}_N \hookrightarrow \mathfrak{osp}_{M|N}$. If we further define

$$\mathcal{C} = \{(i,j) \in (\mathbb{Z}^+)^2 \mid M+1 \leq i \leq M+N,\ 1 \leq j \leq M\},$$

then the union

$$\mathcal{B}_{M|N} = \mathcal{B}^M \cup \mathcal{B}_N \cup \mathcal{C} \tag{2.3.15}$$

indexes a basis $\{F_{ij} \mid (i,j) \in \mathcal{B}_{M|N}\}$ for $\mathfrak{osp}_{M|N}$.

**Corollary 2.3.5.** *The supercenter $\mathrm{ZY}(\mathfrak{osp}_{M|N})$ of $\mathrm{Y}(\mathfrak{osp}_{M|N})$ is trivial: $\mathbb{C}\cdot\mathbf{1}$.*

*Proof.* It is known by [Naz99, Proposition 3.6] that if a Lie superalgebra $\mathfrak{g}$ has trivial supercenter, then so does $\mathfrak{U}(\mathfrak{g}[z])$. As $\mathfrak{osp}_{M|N}$ is simple, Nazarov's result implies that the associated graded $\mathrm{gr}\,\mathrm{Y}(\mathfrak{osp}_{M|N})$ has trivial supercenter by Theorem 2.3.3; hence, the same is true for $\mathrm{Y}(\mathfrak{osp}_{M|N})$ as well. □

**Proposition 2.3.6.** *There is a Hopf superalgebra embedding*

$$\iota\colon \mathfrak{U}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathrm{Y}(\mathfrak{osp}_{M|N}), \quad F_{ij} \mapsto (-1)^{[i]}\mathcal{T}_{ij}^{(1)}. \tag{2.3.16}$$

*for all $1 \leq i,j \leq M+N$.*

*Proof.* Relations (2.2.26) give

$$\begin{aligned}\big[\mathcal{T}_{ij}^{(1)}, \mathcal{T}_{kl}(v)\big] &= \delta_{jk}(-1)^{[j]}\mathcal{T}_{il}(v) - \delta_{il}(-1)^{[i]+([i]+[j])([k]+[l])}\mathcal{T}_{kj}(v) \\ &\quad - \delta_{\bar{\imath}k}(-1)^{[j]+[i][j]+[i]}\theta_i\theta_j\mathcal{T}_{\bar{\jmath}l}(v) + \delta_{\bar{\jmath}l}(-1)^{[j]+([i]+[j])[k]}\theta_i\theta_j\mathcal{T}_{k\bar{\imath}}(v),\end{aligned}$$

for all $1 \le i,j,k,l \le M+N$, so one takes the coefficient of $(-1)^{[i]+[k]}v^{-1}$. Furthermore, equation (2.2.27) gives the relation $\mathcal{T}_{ij}^{(1)} + (-1)^{[i][j]+[j]}\theta_i\theta_j\mathcal{T}_{\bar{\jmath}\bar{\imath}}^{(1)} = 0$ for all indices $1 \le i,j \le M+N$, so we multiply such expression by $(-1)^{[i]}$. Thus, the map is a superalgebra morphism and we observe the Hopf superstructures are compatible, so all that remains to show is injectivity, but this follows from Corollary 2.3.4. □

### 2.3.3 Homogeneous quantization

As the orthosymplectic Lie superalgebra $\mathfrak{g} = \mathfrak{osp}_{M|N}$ is basic, it comes equipped with an even, non-degenerate, super-symmetric, and $\mathfrak{g}$-invariant $\mathbb{C}$-bilinear form which we denote $\psi = (\,\cdot\,,\cdot\,)\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$. As any two such bilinear forms on $\mathfrak{g}$ are proportional, we may take $\psi\colon (X,Y) \mapsto \frac{1}{2}\operatorname{str}(XY)$, where $\operatorname{str}\colon \mathfrak{g} \to \mathbb{C}$ is the super-trace.

Recall that $\mathbb{C} = \mathbb{C}^{1|0}$ is equipped with the trivial $\mathbb{Z}_2$-grading, so the dual space $\mathfrak{g}^*$ is graded as a super vector space $\mathfrak{g}^*_{\bar{0}} \oplus \mathfrak{g}^*_{\bar{1}}$ via the assignment $\mathfrak{g}^*_\gamma = \{\varphi \in \mathfrak{g}^* \mid \varphi(\mathfrak{g}^*_{\gamma+\bar{1}}) = 0\}$ for $\gamma \in \mathbb{Z}_2$. Since the bilinear form $\psi$ is even and non-degenerate, the $\mathbb{C}$-linear maps $\psi_L, \psi_R\colon \mathfrak{g} \to \mathfrak{g}^*$ defined by $\psi_L(v) = \psi(v,\cdot\,)$ and $\psi_R(v) = \psi(\,\cdot\,,v)$, respectively, are super vector space isomorphisms.

Considering the bilinear map $\widetilde{\varkappa}\colon \mathfrak{g}\times\mathfrak{g}^* \to \operatorname{End}(\mathfrak{g})$, $(X,\varphi) \mapsto \varphi_X$ where $\varphi_X$ is the $\mathbb{C}$-linear function $Y \mapsto (-1)^{[\varphi][X]}\varphi(Y)X$ on homogeneous elements $X \in \mathfrak{g}$, $\varphi \in \mathfrak{g}^*$, there is consequently a super vector space isomorphism

$$\varkappa\colon \mathfrak{g}\otimes\mathfrak{g}^* \xrightarrow{\sim} \operatorname{End}(\mathfrak{g}), \quad X\otimes\varphi \mapsto (Y \mapsto (-1)^{[\varphi][X]}\varphi(Y)X)$$

on homogeneous elements $X \in \mathfrak{g}$, $\varphi \in \mathfrak{g}^*$. Indeed, one can confirm $\varkappa$ is a grade preserving and if $\{x_b^*\}_{b=1}^{\dim(\mathfrak{g})} \subset \mathfrak{g}^*$ denotes the dual basis to a homogeneous basis $\mathbf{B} = \{x_b\}_{b=1}^{\dim(\mathfrak{g})}$ for $\mathfrak{g}$, then the inverse of $\varkappa$ is given by $\varkappa^{-1}\colon f \mapsto \sum_{b=1}^{\dim(\mathfrak{g})}(-1)^{[f][x_b]+[x_b]}f(x_b)\otimes x_b^*$.

The *Casimir 2-tensor* $\Omega$ is the preimage of the identity element in $\operatorname{End}(\mathfrak{g})$ under the isomorphism

$$\varkappa\circ(\mathrm{id}\otimes\psi_R)\colon \mathfrak{g}\otimes\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}\otimes\mathfrak{g}^* \xrightarrow{\sim} \operatorname{End}(\mathfrak{g}).$$

Letting $\mathbf{B} = \{x_b\}_{b=1}^{\dim(\mathfrak{g})}$ be some homogeneous basis of $\mathfrak{g}$, with $\mathbf{B}^* = \{y_b\}_{b=1}^{\dim(\mathfrak{g})}$ to denote its dual basis with respect to the bilinear form $\psi$ (so $\psi(x_b, y_c) = \delta_{bc}$ and $\Lambda_b := [x_b] = [y_b]$ for all $1 \leq b, c \leq \dim(\mathfrak{g})$), the Casimir 2-tensor has the form

$$\Omega = \sum_{b=1}^{\dim(\mathfrak{g})} (-1)^{\Lambda_b} x_b \otimes y_b \in \mathfrak{g} \otimes \mathfrak{g}.$$

In terms of the generators (2.1.10), one can compute

$$\frac{1}{2} \operatorname{str}(F_{ij} F_{kl}) = (-1)^{[i]} \delta_{il} \delta_{jk} - (-1)^{[i][j]} \theta_{\bar{\imath}} \theta_{\bar{\jmath}} \delta_{\bar{\imath}k} \delta_{\bar{\jmath}l}$$

for all $1 \leq i, j, k, l \leq M+N$. Thus, given the basis $\{F_{ij} \mid (i,j) \in \mathcal{B}_{M|N}\}$ with $\mathcal{B}_{M|N}$ as in (2.3.15), its dual basis with respect to the form $\psi$ is $\{2^{-\delta_{\bar{\imath}j}}(-1)^{[i]} F_{ji} \mid (i,j) \in \mathcal{B}_{M|N}\}$. The Casimir 2-tensor may therefore be written as

$$\Omega = \sum_{(i,j) \in \mathcal{B}_{M|N}} 2^{-\delta_{\bar{\imath}j}} (-1)^{[j]} F_{ij} \otimes F_{ji} \in \mathfrak{osp}_{M|N} \otimes \mathfrak{osp}_{M|N}.$$

As the bilinear form $\psi$ is super-symmetric and $\mathfrak{g}$-invariant, then the Casimir 2-tensor is so, i.e., $\sigma(\Omega) = \Omega$ where $\sigma$ is the super-braiding and $(\operatorname{ad}_X \otimes \operatorname{id} + \operatorname{id} \otimes \operatorname{ad}_X)(\Omega) = 0$ for all $X \in \mathfrak{g}$ where $\operatorname{ad}_X$ denotes the adjoint action. Furthermore, by identifying $\Omega$ with its image in $\mathfrak{U}(\mathfrak{g})$, the Casimir 2-tensor lies in the supercenter of $\mathfrak{U}(\mathfrak{g})$.

To state the final important property of the Casimir 2-tensor we need, let us first introduce some required notation. Given an element $s \in \mathfrak{g} \otimes \mathfrak{g}$ and indices $1 \leq i < j \leq 3$, we identify elements under the canonical embedding $\mathfrak{g} \hookrightarrow \mathfrak{U}(\mathfrak{g})$ to define a new elements $s_{ij}$ in $\mathfrak{U}(\mathfrak{g})^{\otimes 3}$ via

$$s_{12} := s \otimes 1, \quad s_{23} := 1 \otimes s, \quad \text{and} \quad s_{13} := (\operatorname{id} \otimes \tau)(s \otimes 1),$$

where $\tau \colon \mathfrak{U}(\mathfrak{g})^{\otimes 2} \to \mathfrak{U}(\mathfrak{g})^{\otimes 2}, v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ is the twist map. When $s = s(u, v)$ depends on some formal parameters $u$ and $v$, we shall write $s_{ij}(u_i, u_j)$ for $s(u, v)_{ij}$. If the element $s \in \mathfrak{g} \otimes \mathfrak{g}$ is even, it may be written as a sum of homogeneous decomposable tensors $s = \sum_{n=1}^{d} a_n \otimes b_n$, where $\gamma_n := [a_n] = [b_n]$.

For indices $1 \leq i < j \leq 3$ and $1 \leq k < l \leq 3$, we may consider the commutator $[s_{ij}, s_{kl}] = s_{ij} s_{kl} - s_{kl} s_{ij}$ and use the defining relations of the universal enveloping superalgebra to yield

$$[s_{12}, s_{13}] = \sum_{m,n=1}^{d} (-1)^{\gamma_m\gamma_n}[a_m, a_n] \otimes b_m \otimes b_n, \quad [s_{12}, s_{23}] = \sum_{m,n=1}^{d} a_m \otimes [b_m, a_n] \otimes b_n,$$

$$\text{and} \quad [s_{13}, s_{23}] = \sum_{m,n=1}^{d} (-1)^{\gamma_m\gamma_n} a_m \otimes a_n \otimes [b_m, b_n].$$

In particular, we are able to interpret the above commutators as elements in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

If $r(u, v)$ is a function in formal parameters $u$ and $v$ with coefficients in $(\mathfrak{g} \otimes \mathfrak{g})_{\bar{0}}$, we say that $r(u, v)$ is an *r-matrix* if it satisfies the *super classical Yang-Baxter equation* (SCYBE), i.e., $\mathrm{SCYB}(r(u, v)) = 0$, where

$$\begin{aligned} &\mathrm{SCYB}(r(u, v)) \\ &\quad = [r_{12}(u_1, u_2), r_{13}(u_1, u_3)] + [r_{12}(u_1, u_2), r_{23}(u_2, u_3)] + [r_{13}(u_1, u_3), r_{23}(u_2, u_3)]. \end{aligned}$$

For instance, if $r(u, v) \in (\mathfrak{g} \otimes \mathfrak{g})_{\bar{0}}[\![u^{\pm 1}, v^{\pm 1}]\!]$, then $\mathrm{SCYB}(r(u, v))$ may be regarded as an element in the space $(\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})_{\bar{0}}[\![u_1^{\pm 1}, u_2^{\pm 1}, u_3^{\pm 1}]\!]$. In fact, the $\mathfrak{g}$-invariance of the Casimir 2-tensor implies $\Omega/(u - v)$ is an $r$-matrix:

$$\mathrm{SCYB}(\Omega/(u - v)) = 0.$$

Using the aforementioned properties of the Casimir 2-tensor, one is able to equip the polynomial current superalgebra $\mathfrak{g}[z]$ with a Lie superbialgebra structure $(\mathfrak{g}[z], \delta)$ determined by the Lie co-superbracket

$$\begin{aligned} \delta\colon \mathfrak{g}[z] &\to (\mathfrak{g} \otimes \mathfrak{g})\,[u, v] \cong \mathfrak{g}[z] \otimes \mathfrak{g}[z] \\ f(z) &\mapsto \left(\mathrm{ad}_{f(u)} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{f(v)}\right)\left(\frac{\Omega}{u - v}\right). \end{aligned} \tag{2.3.17}$$

At face value, it is not clear that the above map is well-defied since the element $\Omega/(u-v)$ can not be interpreted as an element in $(\mathfrak{g} \otimes \mathfrak{g})\,[u, v]$. However, using the $\mathfrak{g}$-invariance of $\Omega$, one can prove the above map is equivalent to the assignment

$$\delta(Xz^n) = \sum_{b=1}^{\dim(\mathfrak{g})} \sum_{a=0}^{n-1} (-1)^{\Lambda_b}[X, x_b]z^a \otimes y_b z^{n-a-1}$$

on homogeneous $X \in \mathfrak{g}$ and $n \in \mathbb{Z}^+$, where it is understood that $\delta(X) = 0$. In particular, the defining relations of $\mathfrak{osp}_{M|N}$ show that the Lie co-superbracket is given on generators by the formula

$$\delta(F_{ij}z^n) = \sum_{k=0}^{M+N} (-1)^{[k]} \sum_{a+b=n-1} \left( F_{ik}z^a \otimes F_{kj}z^b - (-1)^{([i]+[k])([k]+[j])} F_{kj}z^a \otimes F_{ik}z^b \right).$$

We now establish terminology relating to deformation and quantization theory of superalgebras over $\mathbb{C}[\hbar]$, where $\hbar$ is a formal element of $\mathbb{Z}_2$-degree $\bar{0}$. To this effect, given any Hopf superalgebra $\mathcal{A}$ over $\mathbb{C}$, a *Hopf superalgebra deformation of* $\mathcal{A}$ (over $\mathbb{C}[\hbar]$) is a Hopf superalgebra $\mathcal{A}_\hbar$ over $\mathbb{C}[\hbar]$ such that:

(i) $\mathcal{A}_\hbar$ is torsion-free as a $\mathbb{C}[\hbar]$-module.

(ii) The quotient $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ is isomorphic to $\mathcal{A}$ as a Hopf superalgebra.

Regarding $\mathbb{C}[\hbar] = \bigoplus_{k\in\mathbb{N}} \mathbb{C}\hbar^k$ as an $\mathbb{N}$-graded ring, such deformation is called *homogeneous* if both $\mathcal{A}$ and $\mathcal{A}_\hbar$ are $\mathbb{N}$-graded modules such that the isomorphism $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \cong \mathcal{A}$ preserves these gradations. A direct super-analogue of [CP95, Proposition 6.2.7] shows that if $\mathbb{U}_\hbar(\mathfrak{b}) = \mathfrak{U}(\mathfrak{b})_\hbar$ is any Hopf superalgebra deformation of $\mathcal{A} = \mathfrak{U}(\mathfrak{b})$ for any Lie superalgebra $\mathfrak{b}$, then $\mathfrak{b}$ is endowed with a Lie superbialgebra structure $(\mathfrak{b}, \delta_\mathfrak{b})$ defined by the Lie co-superbracket

$$\delta_\mathfrak{b}(X) := \frac{\Delta_\hbar(\widetilde{X}) - \Delta_\hbar^{\mathrm{cop}}(\widetilde{X})}{\hbar} \mod \hbar\left(\mathbb{U}_\hbar(\mathfrak{b}) \otimes \mathbb{U}_\hbar(\mathfrak{b})\right) \qquad \text{for all} \quad X \in \mathfrak{b}, \tag{2.3.18}$$

where $\Delta_\hbar$ is the comultiplication map on $\mathbb{U}_\hbar(\mathfrak{b})$, $\Delta_\hbar^{\mathrm{cop}} = \sigma \circ \Delta_\hbar$ is the co-opposite comultiplication, and $\widetilde{X}$ is any element in the fiber of $X \in \mathfrak{b} \subset \mathfrak{U}(\mathfrak{b})$ under the composition $\mathbb{U}_\hbar(\mathfrak{b}) \twoheadrightarrow \mathbb{U}_\hbar(\mathfrak{b})/\hbar\mathbb{U}_\hbar(\mathfrak{b}) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{b})$. Accordingly, a *quantization* of a Lie superbialgebra $(\mathfrak{b}, \delta_\mathfrak{b})$ (over $\mathbb{C}[\hbar]$) is a Hopf superalgebra $\mathbb{U}_\hbar(\mathfrak{b})$ over $\mathbb{C}[\hbar]$ such that:

(i) $\mathbb{U}_\hbar(\mathfrak{b})$ is a Hopf superalgebra deformation of $\mathfrak{U}(\mathfrak{b})$.

(ii) The Lie co-superbracket $\delta_\mathfrak{b}$ coincides with the form (2.3.18).

When the Lie superbialgebra $(\mathfrak{b}, \delta_\mathfrak{b})$ is $\mathbb{N}$-graded, it induces an $\mathbb{N}$-grading on $\mathfrak{U}(\mathfrak{b})$; hence, a quantization $\mathbb{U}_\hbar(\mathfrak{b})$ is called *homogeneous* if the deformation is so.

In the theory of quantum groups, deformations and quantizations are traditionally defined instead as topological Hopf superalgebras over $\mathbb{C}[\![\hbar]\!]$, where the topological tensor product is taken as the $\hbar$-adic completion of the algebraic one (refer to [Dri85], [CP95, §6] for such definitions). However, as discussed in detail in [Wen22], if $\mathbb{U}_\hbar(\mathfrak{b})$ is a homogeneous quantization (over $\mathbb{C}[\hbar]$) of an $\mathbb{N}$-graded Lie superbialgebra $(\mathfrak{b}, \delta_{\mathfrak{b}})$, then its $\hbar$-adic completion

$$\widehat{\mathbb{U}}_\hbar(\mathfrak{b}) = \varprojlim \mathbb{U}_\hbar(\mathfrak{b})/\hbar^n \mathbb{U}_\hbar(\mathfrak{b})$$

will be a homogeneous quantization of $(\mathfrak{b}, \delta_{\mathfrak{b}})$ in the sense of [Dri85], taking into account the super-analogues of the definitions therein. We shall now construct such a homogeneous quantization of $(\mathfrak{g}[z] = \mathfrak{osp}_{M|N}[z], \delta)$, where $\delta$ is the Lie co-superbracket (2.3.17).

**Definition 2.3.7.** Given the tensor product $\mathbb{C}[\hbar] \otimes \mathrm{Y}(\mathfrak{osp}_{M|N}) = \mathrm{Y}(\mathfrak{osp}_{M|N})[\hbar]$ where $\hbar$ is a formal element of $\mathbb{Z}_2$-degree $\bar{0}$, the *Yangian* $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ is defined as the Rees superalgebra of $\mathrm{Y}(\mathfrak{osp}_{M|N})$ with respect to the filtration $\mathbf{F}$ (2.2.28):

$$\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) := R_\hbar(\mathrm{Y}(\mathfrak{osp}_{M|N})) = \bigoplus_{n\in\mathbb{N}} \hbar^n \mathbf{F}_n \subset \mathrm{Y}(\mathfrak{osp}_{M|N})[\hbar].$$

By definition, the Yangian $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ is $\mathbb{N}$-graded and it further comes equipped with a Hopf superstructure by extending the one on $\mathrm{Y}(\mathfrak{osp}_{M|N})$ by $\mathbb{C}[\hbar]$-linearity. In particular, by setting $\widetilde{\mathcal{T}}_{ij}^{(n)} = \hbar^{n-1}\mathcal{T}_{ij}^{(n)}$ for all $1 \leq i, j \leq M+N$ and $n \in \mathbb{Z}^+$, such Hopf superstructure is given by the comultiplication

$$\begin{aligned}\Delta_\hbar\colon\ & \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \to \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \otimes_{\mathbb{C}[\hbar]} \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \\ & \widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto \widetilde{\mathcal{T}}_{ij}^{(n)} \otimes \mathbf{1} + \mathbf{1} \otimes \widetilde{\mathcal{T}}_{ij}^{(n)} + \hbar \sum\nolimits_{k=1}^{M+N} \sum\nolimits_{a=1}^{n-1} \widetilde{\mathcal{T}}_{ik}^{(a)} \otimes \widetilde{\mathcal{T}}_{kj}^{(n-a)},\end{aligned}$$

the counit

$$\varepsilon_\hbar\colon\ \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \to \mathbb{C}[\hbar], \quad \widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto 0,$$

and the antipode

$$\begin{aligned}S_\hbar\colon\ & \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \to \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \\ & \widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto (-1)^{[i][j]+[j]}\theta_i\theta_j \sum\nolimits_{p=1}^{n} \binom{n-1}{n-p} (-\kappa)^{n-p} \hbar^{n-p} \widetilde{\mathcal{T}}_{\bar{j}\bar{i}}^{(p)},\end{aligned}$$

for all $1 \leq i, j \leq M+N$ and $n \in \mathbb{Z}^+$. We now arrive at the main proposition of this subsection:

**Proposition 2.3.8.** *The Yangian* $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ *is a homogeneous quantization of the Lie superbialgebra* $(\mathfrak{osp}_{M|N}[z],\delta)$. *Furthermore, there is a superalgebra isomorphism*

$$\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})/(\hbar-\lambda)\,\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \cong \mathrm{Y}(\mathfrak{osp}_{M|N}) \quad \textit{for all} \quad \lambda\in\mathbb{C}^*.$$

*Proof.* To show that $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ is a homogeneous Hopf superalgebra deformation of the universal enveloping superalgebra $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$, we first observe that $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ is torsion-free, as it is a $\mathbb{C}[\hbar]$-subalgebra of $\mathrm{Y}(\mathfrak{osp}_{M|N})[\hbar]$. Moreover, by composing the Hopf superalgebra isomorphism

$$\phi\colon\ \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})/\hbar\,\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \xrightarrow{\sim} \operatorname{gr}\mathrm{Y}(\mathfrak{osp}_{M|N})$$

mapping

$$\hbar^{n-1}\mathcal{T}_{ij}^{(n)} \ \bmod\ \hbar\,\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \mapsto \overline{\mathcal{T}}_{ij}^{(n)}$$

for $1\le i,j\le M+N$ and $n\in\mathbb{Z}^+$, with the inverse of the isomorphism $\Phi$ (2.3.14), one yields the desired $\mathbb{N}$-graded Hopf superalgebra isomorphism

$$\Phi^{-1}\circ\phi\colon\ \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})/\hbar\,\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{osp}_{M|N}[z]).$$

By the prior discussion, it follows that $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ homogeneously quantizes the Lie superbialgebra structure on $\mathfrak{osp}_{M|N}[z]$ with Lie co-superbracket given by (2.3.18).

We shall show that such Lie co-superbracket coincides with the one given by (2.3.17). Before doing so, we recall that as is the case in Lie bialgebra theory, all Lie superbialgebra structures $(\mathfrak{b},\delta)$ on a Lie superalgebra $\mathfrak{b}$ are in one to one correspondence with coPoisson Hopf superalgebra structures $(\mathfrak{U}(\mathfrak{b}),\delta)$ on $\mathfrak{U}(\mathfrak{b})$.

In particular, any Lie co-superbracket $\delta$ on $\mathfrak{b}$ may be extended to a coPoisson superbracket on $\mathfrak{U}(\mathfrak{b})$, also denoted $\delta$, via the rule

$$\delta(XY)=\delta(X)\Delta(Y)+\Delta(X)\delta(Y) \qquad \text{for all} \quad X,Y\in\mathfrak{b}.$$

Defining $\mathrm{ev}_\hbar$ as the morphism

$$\mathrm{ev}_\hbar\colon\ \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \twoheadrightarrow \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})/\hbar\,\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$$

mapping $\hbar^{n-1}\mathcal{T}_{ij}^{(n)} \mapsto (-1)^{[i]}F_{ij}z^{n-1}$ for $1\le i,j\le M+N$ and $n\in\mathbb{Z}^+$, we obtain the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) & \xrightarrow{\hbar^{-1}(\Delta_\hbar - \Delta_\hbar^{\mathrm{cop}})} & \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})^{\otimes 2} \\
{\scriptstyle \mathrm{ev}_\hbar} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{ev}_\hbar \otimes \mathrm{ev}_\hbar} \\
\mathfrak{U}(\mathfrak{osp}_{M|N}[z]) & \xrightarrow[\delta]{} & \mathfrak{U}(\mathfrak{osp}_{M|N}[z])^{\otimes 2}
\end{array}
$$

where $\delta$ denotes the extension of (2.3.17) to a coPoisson superbracket on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$.

For the second claim, we consider the epimorphism $\mathrm{ev}_\lambda \colon \mathrm{Y}(\mathfrak{osp}_{M|N})[\hbar] \to \mathrm{Y}(\mathfrak{osp}_{M|N})$ induced by the assignment $\hbar \mapsto \lambda$. The restriction $\mathrm{ev}_\lambda^R$ of $\mathrm{ev}_\lambda$ to $R_\hbar(\mathrm{Y}(\mathfrak{osp}_{M|N}))$ will still remain surjective and its kernel is given by

$$\ker(\mathrm{ev}_\lambda^R) = R_\hbar(\mathrm{Y}(\mathfrak{osp}_{M|N})) \cap (\hbar - \lambda)\,\mathrm{Y}(\mathfrak{osp}_{M|N})[\hbar] = (\hbar - \lambda) R_\hbar(\mathrm{Y}(\mathfrak{osp}_{M|N})),$$

proving the proposition. □

As discussed earlier in this subsection, it therefore follows by the work in [Wen22] that the $\hbar$-adic completion

$$\widehat{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) = \varprojlim \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) / \hbar^n\, \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$$

serves as a homogeneous quantization of $(\mathfrak{osp}_{M|N}[z], \delta)$ in the sense of [Dri85]. The remainder of this subsection is devoted to expressing $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ in terms of generators and relations. To do so, we define a new superalgebra $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$ and ultimately prove there exists an isomorphism $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \cong \widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$.

**Definition 2.3.9.** Define $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$ as the unital associative $\mathbb{C}[\hbar]$-superalgebra on the generators $\{\widetilde{\mathcal{T}}_{ij}^{(n)} \mid 1 \le i, j \le M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\big[\widetilde{\mathcal{T}}_{ij}^{(n)}\big] = [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the relations

$$
\begin{aligned}
\big[\widetilde{\mathcal{T}}_{ij}^{(m)}, \widetilde{\mathcal{T}}_{kl}^{(n)}\big] = {} & \delta_{jk}(-1)^{[k]}\, \widetilde{\mathcal{T}}_{il}^{(m+n-1)} - \delta_{il}(-1)^{[i][k]+[j][k]+[j][l]}\, \widetilde{\mathcal{T}}_{kj}^{(m+n-1)} \\
& - \delta_{\bar{\imath}k}(-1)^{[i][j]+[i]+[j]}\theta_i\theta_j \widetilde{\mathcal{T}}_{\bar{\jmath}l}^{(m+n-1)} + \delta_{\bar{\jmath}l}(-1)^{[i][k]+[j][k]+[j]}\theta_i\theta_j \widetilde{\mathcal{T}}_{k\bar{\imath}}^{(m+n-1)} \\
& + (-1)^{[i][j]+[i][k]+[j][k]}\, \hbar \sum_{a=2}^{\min(m,n)} \big(\widetilde{\mathcal{T}}_{kj}^{(a-1)} \widetilde{\mathcal{T}}_{il}^{(m+n-a)} - \widetilde{\mathcal{T}}_{kj}^{(m+n-a)} \widetilde{\mathcal{T}}_{il}^{(a-1)}\big) \\
& + \delta_{\bar{\jmath}l}\, \hbar \sum_{p=1}^{M+N} \sum_{a=2}^{m} \sum_{b=0}^{m-a} \binom{m-a}{b} (\kappa\hbar)^b (-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]} \theta_j\theta_p \widetilde{\mathcal{T}}_{k\bar{p}}^{(m+n-a-b)} \widetilde{\mathcal{T}}_{ip}^{(a-1)} \\
& - \delta_{\bar{\imath}k}\, \hbar \sum_{p=1}^{M+N} \sum_{a=2}^{m} \sum_{b=0}^{m-a} \binom{m-a}{b} (\kappa\hbar)^b (-1)^{[i][j]+[i]+[p]} \theta_i\theta_p \widetilde{\mathcal{T}}_{pj}^{(a-1)} \widetilde{\mathcal{T}}_{\bar{p}l}^{(m+n-a-b)}
\end{aligned}
$$

and

$$\widetilde{\mathcal{T}}_{ij}^{(n)} + (-1)^{[i][j]+[j]}\theta_i\theta_j\widetilde{\mathcal{T}}_{\bar{j}\bar{\imath}}^{(n)} = -\hbar\sum_{p=1}^{M+N}\sum_{a=1}^{n}\sum_{b=1}^{a}\binom{a-1}{a-b}(-\kappa\hbar)^{a-b}(-1)^{[i][p]+[p]}\theta_i\theta_p\widetilde{\mathcal{T}}_{\bar{p}\bar{\imath}}^{(b)}\widetilde{\mathcal{T}}_{pj}^{(n-a)},$$

for all $1 \leq i,j,k,l \leq M+N$ and $m,n \in \mathbb{Z}^+$.

The superalgebra $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$ is $\mathbb{N}$-graded via the gradation assignments

$$\deg\hbar = 1 \quad \text{and} \quad \deg\widetilde{\mathcal{T}}_{ij}^{(n)} = n-1 \quad \text{for} \quad 1 \leq i,j \leq M+N,\ n \in \mathbb{Z}^+.$$

In Proposition 2.3.11 below, it is established that $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) \cong \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$. We note that the following arguments used are similar to those presented in the articles [GRW19a, Proposition 2.2] and [GRW19c, Theorem 6.10].

By equipping $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ with a $\mathbb{C}[\hbar]$-superalgebra structure via the action induced by $\hbar \mapsto 0$, we get the following result:

**Lemma 2.3.10.** *There is an* $\mathbb{N}$*-graded superalgebra epimorphism*

$$\widetilde{\mathrm{ev}}_\hbar\colon \widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) \twoheadrightarrow \mathfrak{U}(\mathfrak{osp}_{M|N}[z]), \quad \widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto (-1)^{[i]}F_{ij}z^{n-1}$$

*for all* $1 \leq i,j \leq M+N$, $n \in \mathbb{Z}^+$. *In particular,* $\ker(\widetilde{\mathrm{ev}}_\hbar) = \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$, *so there is an isomorphism*

$$\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})/\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) \cong \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$$

*as* $\mathbb{N}$*-graded superalgebras.*

*Proof.* By the $\mathbb{C}[\hbar]$-module structure on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$, it is routine to prove $\widetilde{\mathrm{ev}}_\hbar$ is a gradation preserving superalgebra epimorphism such that $\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) \subseteq \ker(\widetilde{\mathrm{ev}}_\hbar)$; hence, $\widetilde{\mathrm{ev}}_\hbar$ descends to an epimorphism $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})/\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) \to \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ of $\mathbb{N}$-graded superalgebras mapping $\widetilde{\mathcal{T}}_{ij}^{(n)} \bmod \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) \mapsto (-1)^{[i]}F_{ij}z^{n-1}$. Conversely, there is a superalgebra morphism $\mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \to \widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})/\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$ sending $F_{ij}z^{n-1} \mapsto (-1)^{[i]}\widetilde{\mathcal{T}}_{ij}^{(n)} \bmod \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$, which establishes the isomorphism. □

**Proposition 2.3.11.** *There is an isomorphism of* $\mathbb{C}[\hbar]$*-superalgebras*

$$\varphi_\hbar \colon \widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) \to \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}), \quad \widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto \hbar^{n-1}\mathcal{T}_{ij}^{(n)}$$

*for all* $1 \le i, j \le M+N$, $n \in \mathbb{Z}^+$.

*Proof.* By the defining relations in the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ and the fact that the elements $\hbar^{n-1}\mathcal{T}_{ij}^{(n)}$, $1 \le i, j \le M+N$, $n \in \mathbb{Z}^+$, generate $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$, it follows that the map $\varphi_\hbar$ is a superalgebra epimorphism.

Recalling the $\mathbb{C}[\hbar]$-superalgebra structure on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ defined by $\hbar \mapsto 0$, there is an epimorphism $\mathrm{ev}_\hbar \colon \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \twoheadrightarrow \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$ of $\mathbb{C}[\hbar]$-superalgebras induced by $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})/\hbar\,\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \cong \mathfrak{U}(\mathfrak{osp}_{M|N}[z])$. In fact, we have the commuting diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N}) & \xrightarrow{\varphi_\hbar} & \mathrm{Y}_\hbar(\mathfrak{osp}_{M|N}) \\ {\scriptstyle \widetilde{\mathrm{ev}}_\hbar}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_\hbar} \\ \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) & \xrightarrow[\mathrm{id}]{} & \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \end{array}$$

Suppose $X \in \widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$ is nonzero such that $X \in \ker \varphi_\hbar$. As there exists a maximal integer $n \in \mathbb{N}$ such that $X \in \hbar^n\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$, one can write $X = \hbar^n Y$ for some element $Y \notin \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$.

In particular, since $0 = \varphi_\hbar(\hbar^n Y) = \hbar^n\varphi_\hbar(Y)$, it must be $Y \in \ker \varphi_\hbar$ as well due to $\mathrm{Y}_\hbar(\mathfrak{osp}_{M|N})$ being torsion-free. However, the above commutative diagram would imply $Y \in \ker(\widetilde{\mathrm{ev}}_\hbar) = \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{osp}_{M|N})$, a contradiction. □

## 2.4 Structure of the Extended Yangian

In this section, we prove many structural results about the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$, including showing that it is isomorphic to the tensor product of the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ with a polynomial algebra in countably many $\mathbb{Z}_2$-grade $\bar{0}$ variables. We will also determine its supercenter and establish a Poincaré-Birkhoff-Witt type theorem for the superalgebra. Broadly, we follow much of the structure of [Wen18, §7], deploying similar arguments to those provided there.

### 2.4.1 The tensor product decomposition, supercenter, and PBW Theorem of $\mathrm{X}(\mathfrak{osp}_{M|N})$

**Definition 2.4.1.** Equipping the polynomial algebra $\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+]$ with the purely even $\mathbb{Z}_2$-grading, the *auxiliary superalgebra* $\mathbb{X}(\mathfrak{osp}_{M|N})$ is the tensor product of $\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+]$ with the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$:

$$\mathbb{X}(\mathfrak{osp}_{M|N}) := \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \otimes \mathrm{Y}(\mathfrak{osp}_{M|N}).$$

Defining $\mathsf{Y}(u) := 1 + \sum_{n=1}^{\infty} \mathsf{y}_n u^{-n} \in (\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+])\,[\![u^{-1}]\!]$, we may consider the following series for $1 \le i, j \le M+N$:

$$\mathsf{T}_{ij}(u) = \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} \mathsf{T}_{ij}^{(n)} u^{-n} := \mathsf{Y}(u) \otimes \mathcal{T}_{ij}(u) \in \mathbb{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!], \tag{2.4.1}$$

with the matrix $\mathsf{T}(u) := \sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes \mathsf{T}_{ij}(u) \in \mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathbb{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]$. Writing $\mathsf{Y}^{[1]}(u)$ for $\mathrm{id} \otimes \mathsf{Y}(u) \otimes \mathbf{1}$ and $\mathcal{T}^{[2]}(u)$ for $\sum_{i,j=1}^{M+N} (-1)^{[i][j]+[j]} E_{ij} \otimes 1 \otimes \mathcal{T}_{ij}(u)$, we may then express $\mathsf{T}(u) = \mathsf{Y}^{[1]}(u)\mathcal{T}^{[2]}(u)$. We equip $\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+]$ with the Hopf algebra structure determined by the comultiplication

$$\Delta_{\mathsf{Y}} \colon \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \to (\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+])^{\otimes 2}, \quad \mathsf{Y}(u) \mapsto \mathsf{Y}(u) \otimes \mathsf{Y}(u),$$

the counit

$$\varepsilon_{\mathsf{Y}} \colon \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \to \mathbb{C}, \quad \mathsf{Y}(u) \mapsto 1,$$

and antipode

$$S_{\mathsf{Y}} \colon \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \to \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+], \quad \mathsf{Y}(u) \mapsto \mathsf{Y}(u)^{-1}.$$

Moreover, we note that since $\mathbb{C}[\mathsf{y}_n \mid n \in \mathbb{Z}^+]$ is a commutative Hopf algebra, its antipode is an involution: $S_{\mathsf{Y}}^2 = \mathrm{id}$. Given the Hopf superstructure maps $\Delta_{\mathrm{Y}}$, $\varepsilon_{\mathrm{Y}}$, $S_{\mathrm{Y}}$ on $\mathrm{Y}(\mathfrak{osp}_{M|N})$, the auxiliary superalgebra $\mathbb{X}(\mathfrak{osp}_{M|N})$ can be equipped with the tensor product Hopf superstructure given by

$$\Delta_{\mathbb{X}} = (\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \circ (\Delta_{\mathsf{Y}} \otimes \Delta_{\mathrm{Y}}), \quad \varepsilon_{\mathbb{X}} = \varepsilon_{\mathsf{Y}} \otimes \varepsilon_{\mathrm{Y}}, \quad S_{\mathbb{X}} = S_{\mathsf{Y}} \otimes S_{\mathrm{Y}},$$

where $\sigma$ is the super-braiding. In particular, such structure maps are given by

$\Delta_{\mathbb{X}}\colon \mathsf{T}(u) \mapsto \mathsf{T}_{[1]}(u)\mathsf{T}_{[2]}(u)$, $\varepsilon_{\mathbb{X}}\colon \mathsf{T}(u) \mapsto \mathbb{1}$, and $S_{\mathbb{X}}\colon \mathsf{T}(u) \mapsto \mathsf{T}(u)^{-1}$, where we note that $\mathsf{T}(u)^{-1} = \mathsf{Y}^{[1]}(u)^{-1}\mathcal{T}^{[2]}(u)^{-1} = \mathcal{T}^{[2]}(u)^{-1}\mathsf{Y}^{[1]}(u)^{-1}$.

By endowing a filtration $\mathbf{H} = \{\mathbf{H}_n\}_{n\in\mathbb{N}}$ on the polynomial algebra $\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+]$ via the filtration degree assignment $\deg_{\mathbf{H}} \mathsf{y}_n = n-1$, we can equip $\mathbb{X}(\mathfrak{osp}_{M|N})$ with the tensor product filtration $\mathbf{F}\big(\mathbb{X}(\mathfrak{osp}_{M|N})\big) = \{\mathbf{F}_n(\mathbb{X}(\mathfrak{osp}_{M|N}))\}_{n\in\mathbb{N}}$ defined by

$$\mathbf{F}_n\big(\mathbb{X}(\mathfrak{osp}_{M|N})\big) = \sum_{a+b=n} \mathbf{H}_a \otimes \mathbf{F}_b\big(\mathrm{Y}(\mathfrak{osp}_{M|N})\big), \tag{2.4.2}$$

where $\{\mathbf{F}_b(\mathrm{Y}(\mathfrak{osp}_{M|N}))\}_{b\in\mathbb{N}}$ is the filtration $\mathbf{F} = \{\mathbf{F}_b\}_{b\in\mathbb{N}}$ on $\mathrm{Y}(\mathfrak{osp}_{M|N})$ as in (2.2.28). Since $\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+]$ is isomorphic to its own associated graded algebra, the mapping $\mathbf{F}_n\big(\mathbb{X}(\mathfrak{osp}_{M|N})\big) \to \bigoplus_{a+b=n} \mathbf{H}_a/\mathbf{H}_{a-1} \otimes \mathbf{F}_b/\mathbf{F}_{b-1}$ induces an isomorphism

$$\mathrm{gr}\,\mathbb{X}(\mathfrak{osp}_{M|N}) \cong \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \otimes \mathrm{gr}\,\mathrm{Y}(\mathfrak{osp}_{M|N}).$$

In particular, by allowing $\overline{\mathsf{T}}_{ij}^{(n)}$ and $\overline{\mathcal{T}}_{ij}^{(n)}$ to denote the respective images of $\mathsf{T}_{ij}^{(n)}$ and $\mathcal{T}_{ij}^{(n)}$ in the $(n-1)^{th}$ graded components of $\mathrm{gr}\,\mathbb{X}(\mathfrak{osp}_{M|N})$ and $\mathrm{gr}\,\mathrm{Y}(\mathfrak{osp}_{M|N})$, identifying the above superalgebras provides

$$\overline{\mathsf{T}}_{ij}^{(n)} = \mathsf{y}_n \otimes \delta_{ij}\mathbf{1} + 1 \otimes \overline{\mathcal{T}}_{ij}^{(n)}$$

since $\mathsf{T}_{ij}^{(n)} = \mathsf{y}_n \otimes \delta_{ij}\mathbf{1} + 1 \otimes \mathcal{T}_{ij}^{(n)} + \sum_{a+b=n} \mathsf{y}_a \otimes \mathcal{T}_{ij}^{(b)}$ for $a, b \in \mathbb{Z}^+$. Given a 1-dimensional abelian Lie superalgebra $\mathfrak{z}_{\mathsf{c}} = \mathbb{C}\cdot\mathsf{c}$ with trivial $\mathbb{Z}_2$-grade, we may use the isomorphism $\psi_{\mathsf{c}}\colon \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \otimes \mathfrak{U}(\mathfrak{osp}_{M|N}[z]) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{osp}_{M|N}[z] \oplus \mathfrak{z}_{\mathsf{c}}[z])$ alongside the inverse $\Phi^{-1}$ of the isomorphism (2.3.14) to construct the isomorphism

$$\psi_{\mathsf{c}} \circ (\mathrm{id} \otimes \Phi^{-1})\colon\ \mathrm{gr}\,\mathbb{X}(\mathfrak{osp}_{M|N}) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{osp}_{M|N}[z] \oplus \mathfrak{z}_{\mathsf{c}}[z]).$$

which sends $\overline{\mathsf{T}}_{ij}^{(n)} \mapsto ((-1)^{[i]}F_{ij} + \delta_{ij}\mathsf{c})z^{n-1}$ for all $1 \le i, j \le M+N$ and $n \in \mathbb{Z}^+$.

**Theorem 2.4.2.** *The assignment $T(u) \mapsto \mathsf{T}(u)$ defines a Hopf superalgebra isomorphism*

$$\chi\colon\ \mathrm{X}(\mathfrak{osp}_{M|N}) \xrightarrow{\sim} \mathbb{X}(\mathfrak{osp}_{M|N}) = \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \otimes \mathrm{Y}(\mathfrak{osp}_{M|N}). \tag{2.4.3}$$

*Proof.* Via the relations (2.2.26) in the Yangian, the map $\chi\colon T(u) \mapsto \mathsf{T}(u)$ defines a morphism of superalgebras; furthermore, we observe the relative Hopf superstructures are compatible. Since $\chi$ preserves the filtrations on $\mathrm{X}(\mathfrak{osp}_{M|N})$ and $\mathbb{X}(\mathfrak{osp}_{M|N})$ given respectively by $\mathbf{E}$ in (2.2.21) and $\mathbf{F}(\mathbb{X}(\mathfrak{osp}_{M|N}))$ via (2.4.2), we can consider the associated graded morphism $\mathrm{gr}\,\chi$. Hence, to show $\chi$ is invertible, it suffices to prove such is true for $\mathrm{gr}\,\chi$.

Given the epimorphism $\Psi\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z] \oplus \mathfrak{z}_{\mathsf{c}}[z]) \to \mathrm{gr}\,\mathrm{X}(\mathfrak{osp}_{M|N})$ as in Proposition 2.2.5, the composition $\mathrm{gr}\,\chi \circ \Psi\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z] \oplus \mathfrak{z}_{\mathsf{c}}[z]) \to \mathrm{gr}\,\mathbb{X}(\mathfrak{osp}_{M|N})$ maps

$$F_{ij}z^{n-1} \mapsto \frac{1}{2}(-1)^{[i]}\big(\overline{\mathsf{T}}_{ij}^{(n)} - (-1)^{[i][j]+[j]}\theta_i\theta_j\overline{\mathsf{T}}_{\bar{j}\bar{i}}^{(n)}\big), \quad \mathsf{c}z^{n-1} \mapsto \frac{1}{2}\big(\overline{\mathsf{T}}_{ii}^{(n)} + \overline{\mathsf{T}}_{\bar{i}\bar{i}}^{(n)}\big),$$

for $1 \le i,j \le M+N$, $n \in \mathbb{Z}^+$. Using the relations (2.1.12), we find that the composition $(\psi_{\mathsf{c}} \circ (\mathrm{id} \otimes \Phi^{-1})) \circ (\mathrm{gr}\,\chi \circ \Psi)$ is equal to the identity map on $\mathfrak{U}(\mathfrak{osp}_{M|N}[z] \oplus \mathfrak{z}_{\mathsf{c}}[z])$; hence, there is an equality $\mathrm{gr}\,\chi \circ \Psi = (\psi_{\mathsf{c}} \circ (\mathrm{id} \otimes \Phi^{-1}))^{-1}$, which implies $\mathrm{gr}\,\chi$ is surjective and $\Psi$ is injective. The existence of $\Psi^{-1}$ means we can write $\mathrm{gr}\,\chi = (\psi_{\mathsf{c}} \circ (\mathrm{id} \otimes \Phi^{-1}))^{-1} \circ \Psi^{-1}$, which proves the invertibility of $\mathrm{gr}\,\chi$. □

Let us define $\mathcal{Y}(u)$ to be the preimage of the series $\mathsf{Y}(u) \otimes \mathbf{1}$ under the isomorphism $\chi$:

$$\mathcal{Y}(u) = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Y}_n u^{-n} := \chi^{-1}\big(\mathsf{Y}(u) \otimes \mathbf{1}\big) \in \mathrm{X}(\mathfrak{osp}_{M|N})[\![u^{-1}]\!]. \tag{2.4.4}$$

Using (2.2.20) where $S_{\mathrm{X}}$ denotes the antipode on $\mathrm{X}(\mathfrak{osp}_{M|N})$, we observe

$$\begin{aligned}\chi(\mathcal{Z}(u)\mathcal{Z}(u+\kappa)^{-1}) = \chi\big(S_{\mathrm{X}}^2(T(u))\big)\chi\big(T(u+2\kappa)^{-1}\big) &= S_{\mathbb{X}}^2(\mathsf{T}(u))\mathsf{T}(u+2\kappa)^{-1} \\ &= \mathsf{Y}(u)\mathsf{Y}(u+2\kappa)^{-1} \otimes \mathbf{1}\end{aligned}$$

since $S_{\mathsf{Y}}^2 = \mathrm{id}$. Thus, via equation (2.2.17) and the computation above, $\chi(\mathcal{Z}(u))$ is given by $\chi(\mathcal{Z}(u)\mathcal{Z}(u+\kappa)^{-1})\chi(\mathcal{Z}(u+\kappa)) = \mathsf{Y}(u)\mathsf{Y}(u+\kappa) \otimes \mathbf{1}$ using that $\mathcal{T}(u)^{-1} = \mathcal{T}(u+\kappa)$. Hence,

$$\mathcal{Z}(u) = \mathcal{Y}(u)\mathcal{Y}(u+\kappa). \tag{2.4.5}$$

**Proposition 2.4.3.** *The collection of elements $\{\mathcal{Y}_n\}_{n\in\mathbb{Z}^+}$ are an algebraically independent set over $\mathbb{C}$ that generates the supercenter of $\mathrm{X}(\mathfrak{osp}_{M|N})$. Consequently, the supercenter of $\mathrm{X}(\mathfrak{osp}_{M|N})$ is*

$$\mathbb{C}[\mathcal{Y}_n \mid n \in \mathbb{Z}^+] \cong \mathrm{ZX}(\mathfrak{osp}_{M|N}) = \mathbb{C}[\mathcal{Z}_n \mid n \in \mathbb{Z}^+].$$

*Proof.* By Corollary 2.3.5, the supercenter of $\mathbb{X}(\mathfrak{osp}_{M|N})$ is $\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \otimes \mathbb{C}\cdot\mathbf{1}$; hence, the collection $\{\chi^{-1}(\mathsf{y}_n \otimes \mathbf{1}) = \mathcal{Y}_n\}_{n\in\mathbb{Z}^+}$ must be an algebraically independent set over $\mathbb{C}$ that generates the supercenter of $\mathrm{X}(\mathfrak{osp}_{M|N})$. Via the relation (2.4.5), it follows that the same must be true for $\{\mathcal{Z}_n\}_{n\in\mathbb{Z}^+}$ as well. □

Through the course of the proof for Theorem 2.4.2, we proved that the epimorphism in Proposition 2.2.5 is injective; thus, we can state the following Poincaré-Birkhoff-Witt-type theorem for the extended Yangian:

**Theorem 2.4.4.** *The epimorphism in Proposition 2.2.5 is an $\mathbb{N}$-graded Hopf superalgebra isomorphism $\Psi\colon \mathfrak{U}(\mathfrak{osp}_{M|N}[z] \oplus \mathfrak{z}_\mathsf{c}[z]) \to \operatorname{gr} \mathrm{X}(\mathfrak{osp}_{M|N})$ given by*

$$F_{ij}z^{n-1} \mapsto (-1)^{[i]}\big(\overline{T}_{ij}^{(n)} - \tfrac{1}{2}\delta_{ij}\overline{\mathcal{Z}}_n\big), \quad \mathsf{c}z^{n-1} \mapsto \tfrac{1}{2}\overline{\mathcal{Z}}_n \tag{2.4.6}$$

*for indices $1 \le i, j \le M+N$, $n \in \mathbb{Z}^+$.*

We now state the explicit form of the Poincaré-Birkhoff-Witt-type theorem for the extended Yangian due to Theorem 2.4.4.

**Corollary 2.4.5** (PBW Theorem for $\mathrm{X}(\mathfrak{osp}_{M|N})$)**.** *Let $\mathcal{B}_{M|N}$ be an index set of pairs $(i,j) \in (\mathbb{Z}^+)^2$ such that $\{F_{ij} \mid (i,j) \in \mathcal{B}_{M|N}\}$ forms a basis for $\mathfrak{osp}_{M|N}$. Given any total ordering '$\preceq$' on the set $\mathsf{X} = \big\{T_{ij}^{(n)}, \mathcal{Z}_r \mid (i,j,n) \in \mathcal{B}_{M|N} \times \mathbb{Z}^+,\ r \in \mathbb{Z}^+\big\}$, the collection of all ordered monomials of the form*

$$X_{n_1}X_{n_2}\cdots X_{n_k}$$

*where $X_{n_a} \in \mathsf{X}$, $X_{n_a} \preceq X_{n_{a+1}}$, and $X_{n_a} \neq X_{n_{a+1}}$ if $X_{n_a}$ is odd, constitutes a basis for the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$.*

*Proof.* As $\{F_{ij}z^{n-1}, \mathsf{c}z^{r-1} \mid (i,j) \in \mathcal{B}_{M|N},\ n,r \in \mathbb{Z}^+\}$ forms a basis for $\mathfrak{osp}_{M|N}[z] \oplus \mathfrak{z}_\mathsf{c}[z]$, then so does the set

$$\big\{(-1)^{[i]}F_{ij}z^{n-1}, ((-1)^{[k]}F_{kk} + \mathsf{c})z^{n-1}, \mathsf{c}z^{r-1} \mid (i,j),(k,k) \in \mathcal{B}_{M|N} : i \neq j;\ n,r \in \mathbb{Z}^+\big\}.$$

□

Furthermore, one can embed $\mathfrak{U}(\mathfrak{osp}_{M|N})$ within the extended Yangian as well:

**Proposition 2.4.6.** *There is a Hopf superalgebra embedding*

$$\iota\colon \mathfrak{U}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathrm{X}(\mathfrak{osp}_{M|N}), \quad F_{ij} \mapsto \frac{1}{2}(-1)^{[i]}\big(T_{ij}^{(1)} - (-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{j}\bar{\imath}}^{(1)}\big) \tag{2.4.7}$$

*for all* $1 \le i,j \le M+N$.

*Proof.* By Theorem 2.4.2, there is an embedding $\iota_Y\colon \mathrm{Y}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathrm{X}(\mathfrak{osp}_{M|N})$ mapping $\mathcal{T}(u) \mapsto \mathcal{Y}(u)^{-1}T(u)$; hence, $\mathcal{T}_{ij}^{(1)} \mapsto T_{ij}^{(1)} - \delta_{ij}\mathcal{Y}_1$ under such inclusion for indices $1 \le i,j \le M+N$.

Using relation (2.4.5) and taking the coefficient of $u^{-1}$ in equation (2.2.18) which yields $T_{ij}^{(1)} + (-1)^{[i][j]+[j]}\theta_i\theta_j T_{ij}^{(1)} = \delta_{ij}\mathcal{Z}_1$, we therefore find $\iota_Y(\mathcal{T}_{ij}^{(1)})$ is equal to the expression $\frac{1}{2}\big(T_{ij}^{(1)} - (-1)^{[i][j]+[j]}\theta_i\theta_j T_{ij}^{(1)}\big)$. Composing the Yangian inclusion $\iota_Y$ with the embedding $\mathfrak{U}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathrm{Y}(\mathfrak{osp}_{M|N})$ in (2.3.16) gives the result. □

## 2.4.2 The Yangian as fixed-point subalgebra of its extended Yangian

As was observed in the proof of Proposition 2.4.6, the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ may be regarded as a Hopf sub-superalgebra of $\mathrm{X}(\mathfrak{osp}_{M|N})$ via the embedding

$$\iota_Y\colon\ \mathrm{Y}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathrm{X}(\mathfrak{osp}_{M|N}), \quad \mathcal{T}(u) \mapsto \mathcal{Y}(u)^{-1}T(u) \tag{2.4.8}$$

which itself is obtained by composing the inverse of the map (2.4.3) with the Hopf superalgebra inclusion $\mathrm{Y}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \otimes \mathrm{Y}(\mathfrak{osp}_{M|N})$. This subsection is dedicated to showing that the image $\iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N}))$ of the Yangian can be realized as fixed point subalgebra of $\mathrm{X}(\mathfrak{osp}_{M|N})$. In particular, we prove the following theorem:

**Theorem 2.4.7.** *The Yangian* $\iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N}))$ *is equal to the subalgebra of* $\mathrm{X}(\mathfrak{osp}_{M|N})$ *fixed by all automorphisms of the form* $\mu_f$ (2.2.9):

$$\iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N})) = \left\{\mathcal{Y} \in \mathrm{X}(\mathfrak{osp}_{M|N}) \mid \mu_f(\mathcal{Y}) = \mathcal{Y} \ \textit{ for all } \ f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]\right\}.$$

*Proof.* We let $\mathrm{X}(\mathfrak{osp}_{M|N})^{\mu_f}$ denote the fixed point subalgebra described by the right hand side of the equation in the theorem statement.

To show the inclusion $\iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N})) \subseteq \mathrm{X}(\mathfrak{osp}_{M|N})^{\mu_f}$, we note that every series $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ defines a superalgebra automorphism $\mu_f^{\mathsf{Y}} \in \mathrm{Aut}(\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+])$ mapping $\mathsf{Y}(u) \mapsto f(u)\mathsf{Y}(u)$, which itself extends to the superalgebra automorphism $\mu_f^{\mathbb{X}} := \mu_f^{\mathsf{Y}} \otimes \mathrm{id} \in \mathrm{Aut}(\mathbb{X}(\mathfrak{osp}_{M|N}))$. Since $\mu_f^{\mathbb{X}}(\mathsf{T}(u)) = f(u)\mathsf{T}(u)$, there is an equality $\chi \circ \mu_f = \mu_f^{\mathbb{X}} \circ \chi$, where $\chi$ is the isomorphism (2.4.3). Hence, $\mu_f(\mathcal{Y}(u)) = f(u)\mathcal{Y}(u)$, which infers $\mu_f(\mathcal{Y}(u)^{-1}T(u)) = \mathcal{Y}(u)^{-1}T(u)$ for all $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$.

For the reverse inclusion $\iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N})) \supseteq \mathrm{X}(\mathfrak{osp}_{M|N})^{\mu_f}$, one can argue similar to [AMR06, Theorem 3.1] by supposing in contradiction that there exists an element $X \in \mathrm{X}(\mathfrak{osp}_{M|N})^{\mu_f} \setminus \iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N}))$. Since the collection of elements $\{\mathcal{Y}_n\}_{n\in\mathbb{Z}^+}$ are central in $\mathrm{X}(\mathfrak{osp}_{M|N})$, there is a decomposition $T(u) = \mathcal{Y}(u)^{-1}T(u)\mathcal{Y}(u)$ which infers every element in $\mathrm{X}(\mathfrak{osp}_{M|N})$ may also be considered as one lying in the polynomial superalgebra $\iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N}))[\mathcal{Y}_n \mid n \in \mathbb{Z}^+]$. In particular, we may write

$$X = B(\mathcal{Y}_1, \dots, \mathcal{Y}_m) \quad \text{for some} \quad B(x_1, \dots, x_m) \in \iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N}))[x_n \mid n \in \mathbb{Z}^+],$$

where $x_n$, $n \in \mathbb{Z}^+$, are indeterminates and $X$ is obtained by evaluating $x_r \mapsto \mathcal{Y}_r$ for $1 \le r \le m$. Note that $B(x_1, \dots, x_m)$ must be non-constant by hypothesis on $X$ and we may further assume $m \in \mathbb{Z}^+$ is minimal in how $X$ may be written in the above form. Expanding $B(x_1, \dots, x_m) = \sum_{p=0}^{d} B_a(x_1, \dots, x_{m-1})x_m^p$ as a polynomial in $x_m$, we write $P(x_m) := \sum_{p=1}^{d} B_a(\mathcal{Y}_1, \dots, \mathcal{Y}_{m-1})x_m^p$ observing that the polynomial degree of $P(x_m)$ must be positive.

For any scalar $a \in \mathbb{C}$, the series $f_a(u) = 1 + au^{-m}$ determines an automorphism $\mu_{f_a}$. Since $\mu_{f_a}(\mathcal{Y}(u)) = f_a(u)\mathcal{Y}(u)$, we note $\mu_{f_a}(\mathcal{Y}_m) = \mathcal{Y}_m + a\mathbf{1}$ and $\mu_{f_a}(\mathcal{Y}_r) = 0$ for all indices $r \neq m$. Using that $\iota_Y(\mathrm{Y}(\mathfrak{osp}_{M|N})) \subseteq \mathrm{X}(\mathfrak{osp}_{M|N})^{\mu_{f_a}}$, it follows

$$X = \mu_{f_a}(X) = B_0(\mathcal{Y}_1, \dots, \mathcal{Y}_{m-1}) + P(\mathcal{Y}_m + a\mathbf{1}) \quad \text{for all} \quad a \in \mathbb{C};$$

hence,

$$P(\mathcal{Y}_m) = P(\mathcal{Y}_m + a\mathbf{1}) \quad \text{for all} \quad a \in \mathbb{C}. \tag{2.4.9}$$

For any $a \in \mathbb{C}$, there is an algebra morphism $\mathrm{ev}_{m,a} \colon \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+] \to \mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+]$ mapping $\mathsf{y}_m \mapsto -a$ and $\mathsf{y}_r \mapsto \mathsf{y}_r$ for all $r \neq m$. Extending $\mathrm{ev}_{m,a}$ to the superalgebra morphism $\mathrm{ev}_{m,a}^{\mathbb{X}} := \mathrm{ev}_{m,a} \otimes \mathrm{id} \colon \mathbb{X}(\mathfrak{osp}_{M|N}) \to \mathbb{X}(\mathfrak{osp}_{M|N})$, we observe

$$\ker(\mathrm{ev}_{m,a}^{\mathbb{X}}) = \ker(\mathrm{ev}_{m,a}) \otimes \mathrm{Y}(\mathfrak{osp}_{M|N}) = (\mathsf{y}_m + a) \otimes \mathrm{Y}(\mathfrak{osp}_{M|N}),$$

where $(\mathsf{y}_m + a)$ is the ideal in $\mathbb{C}[\mathsf{y}_n \,|\, n \in \mathbb{Z}^+]$ generated by the linear polynomial $\mathsf{y}_m + a$. In particular, $\bigcap_{a\in\mathbb{C}} \ker(\mathrm{ev}^{\mathbb{X}}_{m,a})$ is trivial. Since relation (2.4.9) infers $\chi(P(\mathcal{Y}_m)) \in \ker(\mathrm{ev}^{\mathbb{X}}_{m,a})$ for all $a \in \mathbb{C}$ where $\chi$ is the isomorphism (2.4.3), we deduce $\chi(P(\mathcal{Y}_m)) = 0$; thus, $P(\mathcal{Y}_m) = 0$. However, such implies $X = B_0(\mathcal{Y}_1, \ldots, \mathcal{Y}_{m-1})$ contradicting the minimality of $m \in \mathbb{Z}^+$. $\square$

# Chapter 3

# Representation Theory of Orthosymplectic Yangians

After proving many structural results in Chapter 2 on the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ and its extension $\mathrm{X}(\mathfrak{osp}_{M|N})$, we are now equipped with the necessary tools to investigate their representation theories. We note that even if classifications of their finite-dimensional irreducible representations are achieved, they will not canonically lift to ones for all finite-dimensional representations since these latter representation categories are *not* semisimple. As remarked in [CP95, §12.1], the failure of the categories $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}(\mathfrak{osp}_{M|N}))$ and $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N}))$ to be semisimple follows from the failure of $\mathrm{Rep}_{\mathrm{fd}}(\mathfrak{U}(\mathfrak{g}[z]))$ to be so for $\mathfrak{g} = \mathfrak{osp}_{M|N}$ and $\mathfrak{g} = \mathfrak{osp}_{M|N} \oplus \mathfrak{z}_{\mathsf{c}}$, c.f. Theorem 2.3.3 and Theorem 2.4.4. Indeed, for $k \in \mathbb{Z}^+$ and $a \in \mathbb{C}$, the representation of $\mathfrak{g}[z]$ on the space of $k$-jets

$$J_{k,a}(\mathfrak{g}) = \mathfrak{g}[z]/(z-a)^{k+1}\mathfrak{g}[z]$$

is indecomposable but not irreducible as $(z-a)J_{k,a}(\mathfrak{g})$ is a proper submodule. As one will see, Chapter 3 will be heavily focused on the representation theory of the extended Yangian. This is due to the fact that a classification of the finite-dimensional irreducible representations for $\mathrm{X}(\mathfrak{osp}_{M|N})$ will infer a corresponding classification for the Yangian by virtue of the tensor product decomposition (c.f. Theorem 2.4.2):

$$\mathrm{X}(\mathfrak{osp}_{M|N}) \cong \mathbb{C}[\mathcal{Z}_n \mid n \in \mathbb{Z}^+] \otimes \mathrm{Y}(\mathfrak{osp}_{M|N}).$$

The chapter is demarcated in two sizeable sections. The first section § 3.1 lays the foundation for the classification by establishing a highest weight theory for the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ based on fixing certain root systems of $\mathfrak{osp}_{M|N}$ described in §3.1.1. In particular, it is established in §3.1.2 that every finite-dimensional irreducible representation of the extended Yangian is highest weight. Furthermore, the constructions of restriction functors and Verma modules are actualized in subsections § 3.1.3 and § 3.1.4, respectively.

The principal theorems of this chapter lie in the second section §3.2. In §3.2.1, the characterization for the non-triviality of Verma modules is supplied, which allows for the statement of the Theorem 3.2.8 in the following subsection §3.2.2 which describes necessary conditions for all finite-dimensional irreducible representations. To address obtaining sufficient conditions, we follow the strategy of constructing *fundamental* representations, which constitute the subsequent two subsections §3.2.3 and §3.2.4.

## 3.1 Highest Weight Theory of Extended Yangians

In this section, we develop the highest weight theory for the extended Yangian. In fact, we address the construction of two inequivalent highest weight theories on $\mathrm{Rep}(\mathrm{X}(\mathfrak{osp}_{M|N}))$ associated to a selection of two inequivalent positive root systems $\Phi^+_{\mathrm{I}}$ and $\Phi^+_{\mathrm{II}}$ of the orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|N}$. However, for the intents and purposes of the classification of finite-dimensional irreducible representations, one only needs to consider the first highest weight theory; indeed, the subsequent section §3.2 tacitly assumes as such.

Nonetheless, we maintain the more general scheme in this section since both theories infer useful results in and of themselves. For instance, it shown in §3.1.3 that these two highest weight theories respectively prove the non-triviality of covariant functors

$$\begin{aligned} \mathcal{F}^+ \colon\ &\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N})) \to \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{(M-2)|N})) \\ \text{and}\quad \mathcal{F}_+ \colon\ &\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N})) \to \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|(N-2)})). \end{aligned}$$

Furthermore, Proposition 3.1.14 shows that one can always construct non-trivial Verma modules for either highest weight theory.

### 3.1.1 Root systems of orthosymplectic Lie superalgebras

Taking the coefficient of $u^{-1}$ in equation (2.2.18) yields the equality

$$T_{ij}^{(1)} + (-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{j}\bar{\imath}}^{(1)} = \delta_{ij}\mathcal{Z}_1. \tag{3.1.1}$$

Thus, by identifying the generators $F_{ij} \in \mathfrak{osp}_{M|N}$ with their images in $\mathrm{X}(\mathfrak{osp}_{M|N})$ under the embedding (2.4.7), equation (3.1.1) infers $F_{ij} = (-1)^{[i]}T_{ij}^{(1)} - \frac{1}{2}\delta_{ij}(-1)^{[i]}\mathcal{Z}_1$. Under such embedding, the extended Yangian becomes $\mathfrak{osp}_{M|N}$-module and to determine the action of the generators $F_{ij}$ on $\mathrm{X}(\mathfrak{osp}_{M|N})$, we observe that the coefficient of $u^{-1}$ in the defining relations (2.2.8) give

$$\begin{aligned}\big[T_{ij}^{(1)}, T_{kl}(v)\big] &= \delta_{jk}(-1)^{[j]}T_{il}(v) - \delta_{il}(-1)^{[i]+([i]+[j])([k]+[l])}T_{kj}(v)\\ &\quad - \delta_{\bar{\imath}k}(-1)^{[j]+[i][j]+[i]}\theta_i\theta_j T_{\bar{j}l}(v) + \delta_{\bar{j}l}(-1)^{[j]+([i]+[j])[k]}\theta_i\theta_j T_{k\bar{\imath}}(v).\end{aligned} \tag{3.1.2}$$

Hence, since $\mathcal{Z}_1$ lies in the supercenter $\mathrm{ZX}(\mathfrak{osp}_{M|N})$, the generators $F_{ij}$ act on $\mathrm{X}(\mathfrak{osp}_{M|N})$ by the formula

$$\begin{aligned}\big[F_{ij}, T_{kl}(u)\big] &= \delta_{jk}(-1)^{[i]+[j]}T_{il}(u) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}T_{kj}(u)\\ &\quad - \delta_{\bar{\imath}k}(-1)^{[i][j]+[j]}\theta_i\theta_j T_{\bar{j}l}(u) + \delta_{\bar{j}l}(-1)^{([i]+[j])([k]+1)}\theta_i\theta_j T_{k\bar{\imath}}(u).\end{aligned} \tag{3.1.3}$$

Let us set $m = \lfloor \frac{M}{2} \rfloor$, $\widehat{m} = \lceil \frac{M}{2} \rceil$, and $n = \frac{N}{2}$, where $\lfloor\cdot\rfloor \colon \mathbb{Q} \to \mathbb{N}$ denotes the floor function and $\lceil\cdot\rceil \colon \mathbb{Q} \to \mathbb{N}$ denotes the ceiling function. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{osp}_{M|N}$ given by

$$\mathfrak{h} = \bigoplus_{h \in K} \mathbb{C}F_{hh}, \quad \text{where} \quad K = \{1, \ldots, m; M+1, \ldots, M+n\}.$$

Note that $\mathfrak{osp}_{M|N}$ is of rank $m+n$ and the action of the Cartan subalgebra on $\mathrm{X}(\mathfrak{osp}_{M|N})$ is given by

$$\big[F_{hh}, T_{ij}(u)\big] = (\delta_{hi} - \delta_{hj} - \delta_{h\bar{\imath}} + \delta_{h\bar{j}})T_{ij}(u). \tag{3.1.4}$$

Further, let $\{\varepsilon_i, \delta_j \mid 1 \le i \le m,\ 1 \le j \le n\}$ denote the dual basis for $\mathfrak{h}$, where $\varepsilon_i$ and $\delta_j$ are those $\mathbb{C}$-linear functionals in $\mathfrak{h}^*$ given by

$$\varepsilon_i(F_{hh}) = \delta_{ih} \quad \text{and} \quad \delta_j(F_{hh}) = \delta_{M+j,h} \quad \text{for} \quad h \in K.$$

To introduce a notion of positivity for the root system of $\mathfrak{osp}_{M|N}$, we declare the nonzero even generators $F_{ij}$ of $\mathfrak{osp}_{M|N}$ with indices satisfying $i<j$ to be *positive even root vectors*; the collection of their corresponding roots will form a system of positive even roots, which we will denote $\Phi^+_{\text{even}}$.

We will complete the set $\Phi^+_{\text{even}}$ to positive root system of $\mathfrak{osp}_{M|N}$ in two ways by selecting appropriate collections of odd roots to be positive, with the first collection of odd positive roots denoted $\Phi^+_{\text{odd[I]}}$, and the second by $\Phi^+_{\text{odd[II]}}$. The two resulting positive root systems for $\mathfrak{osp}_{M|N}$ will be denoted respectively as

$$\Phi^+_{\text{I}} := \Phi^+_{\text{even}} \cup \Phi^+_{\text{odd[I]}} \quad \text{and} \quad \Phi^+_{\text{II}} := \Phi^+_{\text{even}} \cup \Phi^+_{\text{odd[II]}}.$$

The orthosymplectic Lie superalgebras comprise three infinite families of basic Lie superalgebras: $B(m,n)=\mathfrak{osp}_{(2m+1)|2n}$ for integers $m\geq 0$ and $n\geq 1$; $C(n+1)=\mathfrak{osp}_{2|2n}$ for integers $n\geq 1$; and $D(m,n)=\mathfrak{osp}_{2m|2n}$ for integers $m\geq 2$ and $n\geq 1$. As the descriptions of the root systems of $\mathfrak{osp}_{M|N}$ vary depending on the family, we will describe each of these instances separately. Further, we note that the Cartan subalgebra action on $\mathfrak{osp}_{M|N}$ is described by

$$\big[F_{hh}, F_{ij}\big] = (\delta_{hi} - \delta_{hj} - \delta_{h\bar{\imath}} + \delta_{h\bar{\jmath}})F_{ij}.$$

Considering the case $\mathfrak{osp}_{M|N}=B(m,n)=\mathfrak{osp}_{(2m+1)|2n}$, the collection of its associated positive even roots is given by

$$\Phi^+_{\text{even}} = \{\pm\varepsilon_i - \varepsilon_j,\ -\varepsilon_q,\ \pm\delta_k - \delta_l,\ -2\delta_p\},$$

with $1\leq i<j\leq m$, $1\leq q\leq m$, $1\leq k<l\leq n$, and $1\leq p\leq n$. The first selection of positive odd roots are

$$\Phi^+_{\text{odd[I]}} = \{-\varepsilon_i \pm \delta_k,\ -\delta_p\},$$

where $1\leq i\leq m$ and $1\leq k,p\leq n$, which gives a simple root system $\Delta_{\text{I}}$ with corresponding Dynkin diagram:

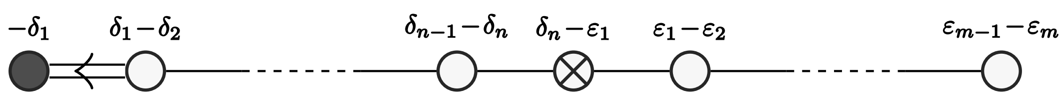


Figure 3.1: Dynkin diagram corresponding to $\Delta_{\text{I}}$ for $B(m,n)$

The second selection of positive odds roots are

$$\Phi^+_{\text{odd[II]}} = \{\pm\varepsilon_i - \delta_k,\ -\delta_p\},$$

when $1 \le i \le m$ and $1 \le k, p \le n$ as well. The simple root system $\Delta_{\text{II}}$ corresponding to this second positive root system has the Dynkin diagram:

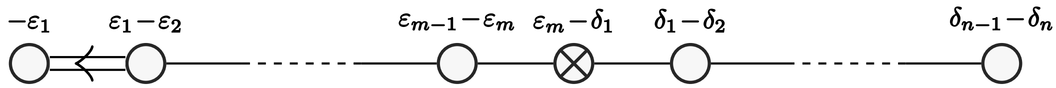


Figure 3.2: Dynkin diagram corresponding to $\Delta_{\text{II}}$ for $B(m, n)$

Here, we observe the black nodes correspond to simple odd roots of nonzero length whereas the tensor nodes correspond to isotropic simple odd roots. Consequently, we remark that the first simple root system is not a distinguished one, in so far as that it has more than one simple odd root. Thus, for a detailed description on how these Dynkin diagrams are constructed in general, we refer the reader to [Zha14].

In the case $\mathfrak{osp}_{M|N} = C(n+1) = \mathfrak{osp}_{2|2n}$, the collection of even positive roots is given by

$$\Phi^+_{\text{even}} = \{\pm\delta_k - \delta_l,\ -2\delta_p\},$$

where $1 \le k < l \le n$ and $1 \le p \le n$. We complete such to a positive root system by selecting the positive odd roots to be

$$\Phi^+_{\text{odd[I]}} = \{-\varepsilon_1 \pm \delta_k\},$$

where $1 \le k \le n$. In such case, the system of simple roots $\Delta_{\text{I}}$ has the Dynkin diagram:


Figure 3.3: Dynkin diagram corresponding to $\Delta_{\text{I}}$ for $C(n+1)$

The second selection of positive odd roots are

$$\Phi^+_{\text{odd[II]}} = \{\pm\varepsilon_1 - \delta_k\},$$

with $1 \le k \le n$. The simple root system $\Delta_{\text{II}}$ corresponding with this second positive root system has the Dynkin diagram:

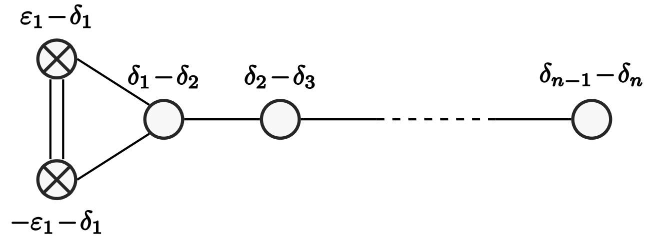


Figure 3.4: Dynkin diagram corresponding to $\Delta_{\text{II}}$ for $C(n+1)$

Lastly, when $\mathfrak{osp}_{M|N} = D(m,n) = \mathfrak{osp}_{2m|2n}$, its collection of even positive roots are

$$\Phi^+_{\text{even}} = \{\pm\varepsilon_i - \varepsilon_j,\ \pm\delta_k - \delta_l,\ -2\delta_p\},$$

where $1 \le i < j \le m$, $1 \le k < l \le n$, and $1 \le p \le n$. To complete such to a set of positive roots, the first selection of positive odd roots will be

$$\Phi^+_{\text{odd[I]}} = \{-\varepsilon_i \pm \delta_k\},$$

where $1 \le i \le m$ and $1 \le k \le n$. The Dynkin diagram corresponding to the simple roots system $\Delta_{\text{I}}$ of these positive roots is therefore given by:

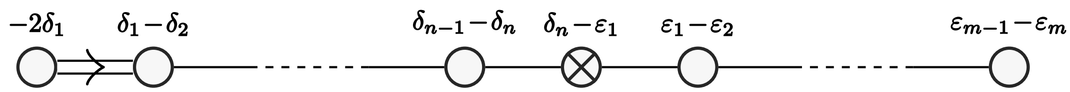


Figure 3.5: Dynkin diagram corresponding to $\Delta_{\text{I}}$ for $D(m,n)$

A second selection of positive odd roots is given as

$$\Phi^+_{\text{odd[II]}} = \{\pm\varepsilon_i - \delta_k\},$$

with $1 \le i \le m$ and $1 \le k \le n$. The simple root system $\Delta_{\text{II}}$ corresponding with this second positive root system has the Dynkin diagram:

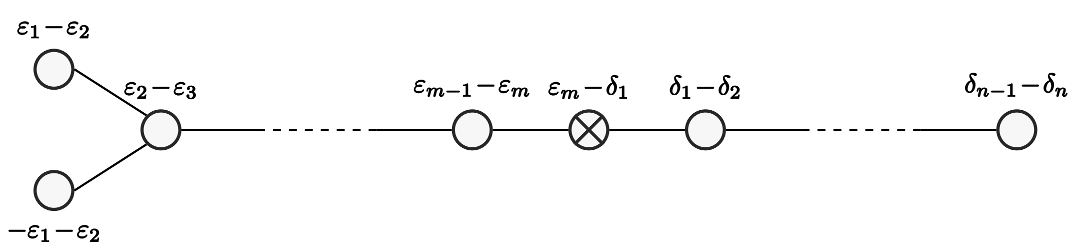


Figure 3.6: Dynkin diagram corresponding to $\Delta_{\text{II}}$ for $D(m,n)$

We will now provide a description of the pairs of indices $(i,j)$ for which the generator $F_{ij}$ will be a positive root vector for either positive root system $\Phi_{\mathrm{I}}^{+}$ or $\Phi_{\mathrm{II}}^{+}$. Setting $\mathbb{Z}_{M+N}^{+} := [1, M+N] \cap \mathbb{Z}^{+}$, we consider the following subsets of $(\mathbb{Z}_{M+N}^{+})^2$:

$$\Gamma_{0,0} = \{(i,j) \mid 1 \le i < j \le M\}\,, \quad \Gamma_{1,1} = \{(i,j) \mid M+1 \le i < j \le M+N\}\,,$$

$$\begin{aligned} \Gamma_{0,1a}^{\mathrm{I}} &= \{(i,j) \mid 1 \le i \le m,\ M+1 \le j \le M+n\}\,,\\ \Gamma_{0,1b}^{\mathrm{I}} &= \{(i,j) \mid 1 \le i \le \widehat{m},\ M+n+1 \le j \le M+N\}\,,\\ \Gamma_{1,0a}^{\mathrm{I}} &= \{(i,j) \mid M+1 \le i \le M+n,\ m+1 \le j \le M\}\,,\\ \Gamma_{1,0b}^{\mathrm{I}} &= \{(i,j) \mid M+n+1 \le i \le M+N,\ \widehat{m}+1 \le j \le M\}\,,\end{aligned}$$

$$\begin{aligned} \Gamma_{0,1}^{\mathrm{II}} &= \{(i,j) \mid 1 \le i \le M,\ M+n+1 \le j \le M+N\}\,,\\ \Gamma_{1,0}^{\mathrm{II}} &= \{(i,j) \mid M+1 \le i \le M+n,\ 1 \le j \le M\}\,,\end{aligned}$$

and assign $\Gamma_{0,1}^{\mathrm{I}} = \Gamma_{0,1a}^{\mathrm{I}} \cup \Gamma_{0,1b}^{\mathrm{I}}$, $\Gamma_{1,0}^{\mathrm{I}} = \Gamma_{1,0a}^{\mathrm{I}} \cup \Gamma_{1,0b}^{\mathrm{I}}$ to define

$$\Gamma_{\mathrm{even}} := \Gamma_{0,0} \cup \Gamma_{1,1}, \quad \Gamma_{\mathrm{odd[I]}} := \Gamma_{0,1}^{\mathrm{I}} \cup \Gamma_{1,0}^{\mathrm{I}}, \quad \Gamma_{\mathrm{odd[II]}} := \Gamma_{0,1}^{\mathrm{II}} \cup \Gamma_{1,0}^{\mathrm{II}}.$$

At last, we can finally define the following sets

$$\Lambda_{\mathrm{I}}^{+} := \Gamma_{\mathrm{even}} \cup \Gamma_{\mathrm{odd[I]}}, \qquad \Lambda_{\mathrm{II}}^{+} := \Gamma_{\mathrm{even}} \cup \Gamma_{\mathrm{odd[II]}}. \tag{3.1.5}$$

Visually, by regarding $(\mathbb{Z}_{M+N}^{+})^2$ as an index set for the entries of an $(M+N) \times (M+N)$ matrix, then those indices that correspond to the dark grey regions of the following diagrams are exactly those that occur in $\Lambda_{\mathrm{I}}^{+}$ and $\Lambda_{\mathrm{II}}^{+}$, respectively:

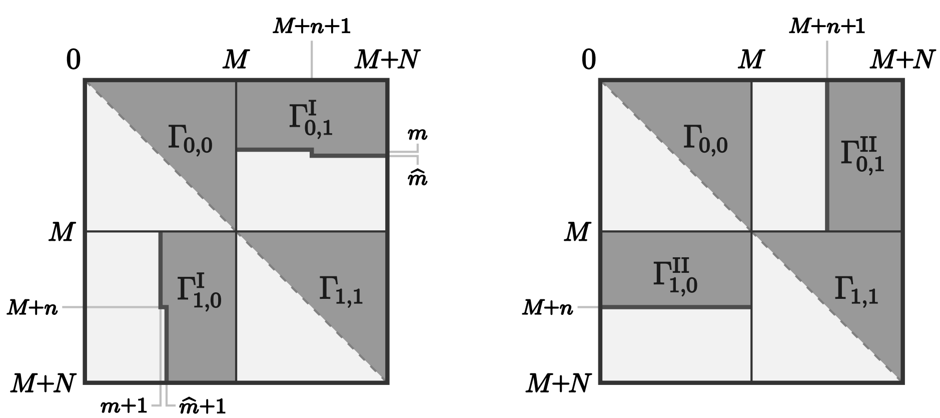


Figure 3.7: Visualizations of $\Lambda_{\mathrm{I}}^{+}$ and $\Lambda_{\mathrm{II}}^{+}$

Further, let us consider the sets

$$\Lambda^{\circ} := \{(i,i) \mid i \in \mathbb{Z}^{+}_{M+N}\}, \tag{3.1.6}$$

$$\Lambda^{-}_{\mathrm{I}} := (\mathbb{Z}^{+}_{M+N})^2 \setminus (\Lambda^{+}_{\mathrm{I}} \cup \Lambda^{\circ}), \quad \text{and} \quad \Lambda^{-}_{\mathrm{II}} := (\mathbb{Z}^{+}_{M+N})^2 \setminus (\Lambda^{+}_{\mathrm{II}} \cup \Lambda^{\circ}). \tag{3.1.7}$$

If we let $\Theta$ denote either I or II, then we will have the triangular decomposition $\mathfrak{osp}_{M|N} = \mathfrak{n}^{-}_{\Theta} \oplus \mathfrak{b}_{\Theta} = \mathfrak{n}^{-}_{\Theta} \oplus \mathfrak{h} \oplus \mathfrak{n}^{+}_{\Theta}$, where $\mathfrak{h}$ is the Cartan subalgebra, $\mathfrak{b}_{\Theta} = \mathfrak{h} \oplus \mathfrak{n}^{+}_{\Theta}$ is the Borel subalgebra, and

$$\mathfrak{n}^{-}_{\Theta} = \mathrm{span}_{\mathbb{C}} \{F_{ij}\}_{(i,j)\in\Lambda^{-}_{\Theta}}, \quad \mathfrak{n}^{+}_{\Theta} = \mathrm{span}_{\mathbb{C}} \{F_{ij}\}_{(i,j)\in\Lambda^{+}_{\Theta}}.$$

Letting $(\mathfrak{osp}_{M|N})_{\alpha} = \{X \in \mathfrak{osp}_{M|N} \mid [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ denote the root space corresponding to $\alpha \in \mathfrak{h}^{*}$, then

$$\mathfrak{n}^{-}_{\Theta} = \bigoplus_{\alpha\in\Phi^{-}_{\Theta}} (\mathfrak{osp}_{M|N})_{\alpha} \quad \text{and} \quad \mathfrak{n}^{+}_{\Theta} = \bigoplus_{\alpha\in\Phi^{+}_{\Theta}} (\mathfrak{osp}_{M|N})_{\alpha}.$$

Furthermore, the action of the Cartan subalgebra on $\mathrm{X}(\mathfrak{osp}_{M|N})$ described by (3.1.4) results in the following decomposition for the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ in terms of the root lattice $\mathbb{Z}\Phi_{\Theta}$:

$$\mathrm{X}(\mathfrak{osp}_{M|N}) = \bigoplus_{\alpha\in\mathbb{Z}\Phi_{\Theta}} \mathrm{X}(\mathfrak{osp}_{M|N})_{\alpha},$$

where $\mathrm{X}(\mathfrak{osp}_{M|N})_{\alpha} = \{X \in \mathrm{X}(\mathfrak{osp}_{M|N}) \mid [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ is the root lattice space for $\alpha \in \mathbb{Z}\Phi_{\Theta}$.

### 3.1.2 Highest weight theory

Via the embedding (2.4.7), any representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ can be pulled back to one for the Lie superalgebra $\mathfrak{osp}_{M|N}$. Therefore, we have the familiar notions of weights and weight vectors for representations $V$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$: for any functional $\mu \in \mathfrak{h}^{*}$, provided

$$V_{\mu} := \{v \in V \mid H \cdot v = \mu(H)v \text{ for all } H \in \mathfrak{h}\} \neq 0,$$

then $\mu$ is called a *weight*, $V_{\mu}$ is called a *weight space*, and nonzero vectors in $V_{\mu}$ are called *weight vectors*. Selecting a system of positive roots $\Phi^{+}_{\mathrm{I}}$ or $\Phi^{+}_{\mathrm{II}}$, we can endow a partial ordering '$\preccurlyeq$' on the set of weights of $V$ via the rule: $\omega \preccurlyeq \mu \Leftrightarrow \mu - \omega$ is an $\mathbb{N}$-linear combination of positive roots of $\mathfrak{osp}_{M|N}$. Furthermore, since $\mathrm{X}(\mathfrak{osp}_{M|N})_\alpha(V_\mu) \subseteq V_{\mu+\alpha}$, then

$$\mathrm{X}(\mathfrak{osp}_{M|N})_\alpha\Big(\bigoplus\nolimits_{\mu\in\mathfrak{h}^*} V_\mu\Big) \subseteq \bigoplus\nolimits_{\mu\in\mathfrak{h}^*} V_\mu. \tag{3.1.8}$$

We now introduce the notion of highest weight:

**Definition 3.1.1.** Let $\Theta$ denote either I or II. A representation $V$ of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ is an $X_\Theta$*-highest weight representation* if there exists a nonzero vector $\xi \in V$ such that $\mathrm{X}(\mathfrak{osp}_{M|N})\xi = V$, and

$$\begin{aligned} T_{ij}(u)\xi &= 0 \qquad \text{for all} \quad (i,j) \in \Lambda_\Theta^+ \\ \text{and} \quad T_{kk}(u)\xi &= \lambda_k(u)\xi \qquad \text{for all} \quad 1 \le k \le M+N, \end{aligned} \tag{3.1.9}$$

where $\lambda_k(u)$ is some formal series

$$\lambda_k(u) = 1 + \sum_{n=1}^{\infty} \lambda_k^{(n)} u^{-n} \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]. \tag{3.1.10}$$

We say that $\xi$ is the $X_\Theta$*-highest weight vector* of $V$ and call the tuple $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ of formal series as the $X_\Theta$*-highest weight* of $V$.

The first main result to prove in a highest weight theory is to show that every finite-dimensional irreducible representation is highest weight. However, to do so in our setting will require proving a rather computational lemma.

We note that many of the techniques used in the proof of the lemma below, and the subsequent theorem, arise from those used in the proof of [AMR06, Theorem 5.1].

**Lemma 3.1.2.** *Let $\Theta$ denote either* I *or* II*. If $\mathcal{I}_\Theta$ is the left graded ideal of* $\mathrm{X}(\mathfrak{osp}_{M|N})$ *generated by the coefficients of $T_{ij}(u)$ for $(i,j) \in \Lambda_\Theta^+$, then*

(i) *for all $(i,j) \in \Lambda_\Theta^+$ and $1 \le k \le M+N$:*

$$T_{ij}(u)T_{kk}(v) \equiv 0 \mod \mathcal{I}_\Theta, \tag{3.1.11}$$

(ii) *for all $1 \le k, l \le M+N$:*

$$\big[T_{kk}(u), T_{ll}(v)\big] \equiv 0 \mod \mathcal{I}_\Theta. \tag{3.1.12}$$

*Proof.* For brevity, we shall only use '$\equiv$' to denote equivalence of elements in $\mathrm{X}(\mathfrak{osp}_{M|N})$ modulo $\mathcal{I}_\Theta$.

(i) We shall provide a proof for $\Theta = \mathrm{I}$ as the case $\Theta = \mathrm{II}$ is similar; accordingly, throughout the proof we shall suppose $(i,j) \in \Lambda_{\mathrm{I}}^+$ and $1 \le k \le M+N$. Throughout, we shall suppose $m = \lfloor \frac{M}{2} \rfloor$, $\widehat{m} = \lceil \frac{M}{2} \rceil$, and $n = \frac{N}{2}$.

To verify equation (3.1.11), we demarcate the problem into the two situations when $(i,k) \in \Lambda_{\mathrm{I}}^+$ and when $(i,k) \notin \Lambda_{\mathrm{I}}^+$. If $(i,k) \in \Lambda_{\mathrm{I}}^+$ such that $k \neq \bar{\imath}$, $k \neq \bar{\jmath}$, then the relation is immediate from $T_{ij}(u)T_{kk}(v) \equiv [T_{ij}(u), T_{kk}(v)]$. However, if $(i,k) \notin \Lambda_{\mathrm{I}}^+$, then $(k,j) \in \Lambda_{\mathrm{I}}^+$, so if we further assume $k \neq \bar{\imath}$, $k \neq \bar{\jmath}$, then the relation follows from $T_{ij}(u)T_{kk}(v) \equiv -[T_{kk}(v), T_{ij}(u)] \equiv 0$. Therefore, the remainder of the proof will be demarcated into four steps to show that equation (3.1.11) is true in the exceptional cases when $k = \bar{\imath}$ or $k = \bar{\jmath}$.

*Step 1.* Let us suppose $(i,k) \in \Lambda_{\mathrm{I}}^+$ and $k = \bar{\imath}$. These conditions necessarily imply that either $1 \le i \le m$ or $M+1 \le i \le M+n$.

When $1 \le i \le m$, we must have $\widehat{m}+1 \le k \le M$ and $(i,j) \in \Gamma_{0,0} \cup \Gamma_{0,1}^{\mathrm{I}}$. Because $j \neq i = \bar{k}$ and $(\bar{k}, k) \in \Gamma_{0,0}$, the defining relations (2.2.8) imply

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{p=1}^{\bar{k}} (-1)^{[k][j]+[k]+[j][p]} \theta_{\bar{k}} \theta_p T_{pj}(u) T_{\bar{p}k}(v).$$

Since $\bar{k} < j$, each index $1 \le p \le \bar{k} \le m$ satisfies $(p,j) \in \Gamma_{0,0} \cup \Gamma_{0,1}^{\mathrm{I}}$; thus, because we have $(p,k) \in \Gamma_{0,0}$ and $k \neq \bar{\jmath}$, for such indices we can compute

$$T_{pj}(u)T_{\bar{p}k}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{q=1}^{\bar{k}} (-1)^{[p][j]+[p]+[j][q]} \theta_p \theta_q T_{qj}(u) T_{\bar{q}k}(v)$$

and therefore, $(-1)^{[p][j]+[p]} \theta_p T_{pj}(u) T_{\bar{p}k}(v) \equiv (-1)^{[k][j]+[k]} \theta_{\bar{k}} T_{\bar{k}j}(u) T_{kk}(v)$. The original equation therefore implies $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

When $M+1 \le i \le M+n$, we have $M+n+1 \le k \le M+N$ and $(i,j) \in \Gamma_{1,0a}^{\mathrm{I}} \cup \Gamma_{1,1}$. Since $j \neq i = \bar{k}$ and $(\bar{k}, k) \in \Gamma_{1,1}$, the defining relations give

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1 \le p \le m, \\ M+1 \le p \le \bar{k}}} (-1)^{[k][j]+[k]+[j][p]} \theta_{\bar{k}} \theta_p T_{pj}(u) T_{\bar{p}k}(v).$$

Note that if $1 \le p \le m$, then $(p,j) \in \Gamma_{0,0} \cup \Gamma^{\mathrm{I}}_{0,1}$; whilst for $M+1 \le p \le \bar{k}$, we have $(p,j) \in \Gamma^{\mathrm{I}}_{1,0a}$ if $(i,j) \in \Gamma^{\mathrm{I}}_{1,0a}$ and $(p,j) \in \Gamma_{1,1}$ if $(i,j) \in \Gamma_{1,1}$ since $p \le \bar{k} = i < j$. For these indices, we can then compute

$$T_{pj}(u)T_{\bar{p}k}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1\le q\le m,\\ M+1\le q\le \bar{k}}} (-1)^{[p][j]+[p]+[j][q]}\theta_p\theta_q T_{qj}(u)T_{\bar{q}k}(v).$$

Therefore, by using a similar argument as before we can deduce $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

*Step 2.* Now suppose $(i,k) \in \Lambda^+_{\mathrm{I}}$ and $k = \bar{\jmath}$. Necessarily, these conditions impose that either $(i,j) \in \Gamma^{\mathrm{I}}_{0,1a}$ or $(i,j) \in \Gamma^{\mathrm{I}}_{0,1b}$, $i \le m$.

When $(i,j) \in \Gamma^{\mathrm{I}}_{0,1a}$, we have $M+1 \le k \le M+n$. Since $(k,\bar{k}) \in \Gamma_{1,1}$ and $k \ne \bar{\imath}$, the relation $T_{i\bar{k}}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{i\bar{k}}(u)\big]$ infers

$$T_{i\bar{k}}(u)T_{kk}(v) \equiv -\frac{1}{v-u-\kappa} \sum_{\substack{1\le p\le m,\\ M+1\le p\le k}} (-1)^{[k]+[k][p]+[p]}\theta_k\theta_p T_{i\bar{p}}(u)T_{kp}(v). \tag{3.1.13}$$

For $1 \le p \le m$, we have $(i,\bar{p}) \in \Gamma_{0,0}$ and $(k,\bar{p}) \in \Gamma^{\mathrm{I}}_{1,0a}$; whilst $M+1 \le p \le k$ satisfy $(i,\bar{p}) \in \Gamma^{\mathrm{I}}_{0,1b}$ and $(k,\bar{p}) \in \Gamma_{1,1}$. Hence, $T_{i\bar{p}}(u)T_{kp}(v) \equiv -\big[T_{kp}(v), T_{i\bar{p}}(u)\big]$ gives

$$T_{i\bar{p}}(u)T_{kp}(v) \equiv -\frac{1}{v-u-\kappa} \sum_{\substack{1\le q\le m,\\ M+1\le q\le k}} (-1)^{[k][i]+[p][i]+[p]+[k][q]+[q]}\theta_p\theta_q T_{i\bar{q}}(u)T_{kq}(v), \tag{3.1.14}$$

and therefore, $(-1)^{[k][i]+[p][i]+[p]}\theta_p T_{i\bar{p}}(u)T_{kp}(v) \equiv (-1)^{[k]}\theta_k T_{i\bar{k}}(u)T_{kk}(v)$. Equation (3.1.13) therefore implies $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

Otherwise, when $(i,j) \in \Gamma^{\mathrm{I}}_{0,1b}$ and $i \le m$, we have $M+n+1 \le k \le M+N$. Since $(i,k) \in \Gamma^{\mathrm{I}}_{0,1b}$ and $k \ne \bar{\imath}$, the defining relations imply

$$T_{i\bar{k}}(u)T_{kk}(v) \equiv \frac{1}{u-v-\kappa} \sum_{p=1}^{i} (-1)^{[i][k]+[i][p]+[p]}\theta_{\bar{k}}\theta_p T_{k\bar{p}}(v)T_{ip}(u). \tag{3.1.15}$$

Now, for each index $1 \le p \le i \le m$ we have $(k,\bar{p}) \in \Gamma^{\mathrm{I}}_{1,0b}$ and $(i,\bar{p}) \in \Gamma_{0,0}$. Since $k \ne \bar{\imath}$, we use $T_{k\bar{p}}(v)T_{ip}(u) \equiv -\big[T_{ip}(u), T_{k\bar{p}}(v)\big]$ to deduce

$$T_{k\bar{p}}(v)T_{ip}(u) \equiv -\frac{1}{u-v-\kappa} \sum_{q=1}^{i} (-1)^{[i][k]+[p][k]+[p]+[i][q]+[q]}\theta_p\theta_q T_{k\bar{q}}(v)T_{iq}(u). \tag{3.1.16}$$

Hence, $-\theta_{\bar{k}} T_{i\bar{k}}(u)T_{kk}(v) \equiv (-1)^{[p][k]+[p]}\theta_p T_{k\bar{p}}(v)T_{ip}(u)$ and so equation (3.1.15) implies $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

*Step 3.* Here, we assume $(i,k) \notin \Lambda_{\mathrm{I}}^{+}$ and $k = \bar{\imath}$. These conditions require that either $\widehat{m} \le i \le M$ or $M+n+1 \le i \le M+N$.

When $\widehat{m} \le i \le M$, we must have either $(i,j) \in \Gamma_{0,0}$ or $(i,j) \in \Gamma_{0,1b}^{\mathrm{I}}$ when $i = \widehat{m}$ and $M$ is odd. In the first case, we therefore have $1 \le k \le \widehat{m}$, $(k,j) \in \Gamma_{0,0}$, and $k \ne \bar{\jmath}$ since $k = \bar{\imath}$. Hence, the defining relations (2.2.8) imply

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{\bar{k}j}(u)\big] \equiv \frac{1}{v-u-\kappa}\sum_{p=1}^{\bar{\jmath}}(-1)^{[k][p]}\theta_k\theta_p T_{pk}(v)T_{\bar{p}j}(u). \quad (3.1.17)$$

For each index $1 \le p \le \bar{\jmath}$, since $\widehat{m} \le i < j \le M$ and $k = \bar{\imath}$, then both $(p,k)$ and $(p,j)$ lie in $\Gamma_{0,0}$. Hence,

$$T_{pk}(v)T_{\bar{p}j}(u) \equiv -\frac{1}{v-u-\kappa}\sum_{q=1}^{\bar{\jmath}}(-1)^{[p][k]+[p]+[k][q]}\theta_p\theta_q T_{qk}(v)T_{\bar{q}j}(u), \quad (3.1.18)$$

and therefore $-\theta_k T_{\bar{k}j}(u)T_{kk}(v) \equiv (-1)^{[p][k]+[p]}\theta_p T_{pk}(v)T_{\bar{p}j}(u)$, implying $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

In the second case, $i = \bar{k} = k = \widehat{m}$ and $(k,j) \in \Gamma_{1,0b}^{\mathrm{I}}$, so we use the equivalence $T_{\bar{k}j}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{\bar{k}j}(u)\big]$ to obtain

$$T_{\bar{k}j}(u)T_{kk}(v) \equiv \frac{1}{v-u-\kappa}\sum_{\substack{1\le p\le m,\\ M+1\le p\le \bar{\jmath}}}(-1)^{[k][p]}\theta_k\theta_p T_{pk}(v)T_{\bar{p}j}(u). \quad (3.1.19)$$

Each index $1 \le p \le m$, satisfies $(p,k) \in \Gamma_{0,0}$ (since $M$ is odd) and $(p,j) \in \Gamma_{0,1b}^{\mathrm{I}}$; whilst each index $M+1 \le p \le \bar{\jmath}$ satisfies $(p,k) \in \Gamma_{1,0a}^{\mathrm{I}}$ and $(p,j) \in \Gamma_{1,1}$. For such indices,

$$T_{pk}(v)T_{\bar{p}j}(u) \equiv -\frac{1}{v-u-\kappa}\sum_{\substack{1\le q\le m,\\ M+1\le q\le \bar{\jmath}}}(-1)^{[p][k]+[p]+[k][q]}\theta_p\theta_q T_{qk}(v)T_{\bar{q}j}(u), \quad (3.1.20)$$

and a similar argument to before shows $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

When $M+n+1 \le i \le M+N$, we have $(i,j) \in \Gamma_{1,0b}^{\mathrm{I}} \cup \Gamma_{1,1}$ and $M+1 \le k \le M+n$. If we first assume $(i,j) \in \Gamma_{1,0b}^{\mathrm{I}}$, then $(k,j) \in \Gamma_{1,0a}^{\mathrm{I}}$. Consequently, we yield the equivalence (3.1.17). Since for each index $1 \le p \le \bar{\jmath}$, we find that $(p,k) \in \Gamma_{0,1a}^{\mathrm{I}}$ and $(p,j) \in \Gamma_{0,0}$, we also get equation (3.1.18) and can therefore deduce $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

If instead $(i, j) \in \Gamma_{1,1}$, then $(k, j) \in \Gamma_{1,1}$, so we can deduce (3.1.19). Each index $1 \leq p \leq m$ satisfies $(p, k) \in \Gamma^{\mathrm{I}}_{0,1a}$ and $(p, j) \in \Gamma^{\mathrm{I}}_{0,1b}$; whilst for each index $M+1 \leq p \leq \bar{\jmath}$, both $(p, k)$ and $(p, j)$ lie in $\Gamma_{1,1}$ since $M+n+1 \leq \bar{k} = i < j \leq M+N$. Therefore, we get the equation (3.1.20) and hence $T_{\bar{k}j}(u)T_{kk}(v) \equiv 0$.

*Step 4.* Finally, we consider the case when $(i, k) \notin \Lambda^+_{\mathrm{I}}$ and $k = \bar{\jmath}$. Necessarily, these conditions imply that either $\widehat{m} \leq i \leq M$, $(i, j) \in \Gamma^{\mathrm{I}}_{1,0}$ where $\widehat{m}+1 \leq j$, or $M+n+1 \leq i < j \leq M+N$.

When $\widehat{m} \leq i \leq M$, it must be that either $(i, j) \in \Gamma_{0,0}$ or $(i, j) \in \Gamma^{\mathrm{I}}_{0,1b}$ when $i = \widehat{m}$ and $M$ is odd. In the first case, we therefore have $1 \leq k \leq m$ and $(k, \bar{k}) \in \Gamma_{0,0}$, so the relation $T_{i\bar{k}}(u)T_{kk}(v) \equiv -\big[T_{kk}(v), T_{i\bar{k}}(u)\big]$ yields

$$T_{i\bar{k}}(u)T_{kk}(v) \equiv -\frac{1}{v-u-\kappa}\sum_{p=1}^{k}(-1)^{[k]+[k][p]+[p]}\theta_k\theta_p T_{i\bar{p}}(u)T_{kp}(v). \tag{3.1.21}$$

Since for indices $1 \leq p \leq k$, we have both $(i, \bar{p})$ and $(k, \bar{p})$ in $\Gamma_{0,0}$, we can use the relation $T_{i\bar{p}}(u)T_{kp}(v) \equiv -\big[T_{kp}(v), T_{i\bar{p}}(u)\big]$ to compute

$$T_{i\bar{p}}(u)T_{kp}(v) \equiv -\frac{1}{v-u-\kappa}\sum_{q=1}^{k}(-1)^{[k][i]+[p][i]+[p]+[k][q]+[q]}\theta_p\theta_q T_{i\bar{q}}(u)T_{kq}(v), \tag{3.1.22}$$

and therefore, $(-1)^{[k][i]+[p][i]+[p]}\theta_p T_{i\bar{p}}(u)T_{kp}(v) \equiv (-1)^{[k]}\theta_k T_{i\bar{k}}(u)T_{kk}(v)$. Equation (3.1.21) therefore implies $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

In the second case we have $M+1 \leq k \leq M+n$, so $(k, \bar{k}) \in \Gamma_{1,1}$ and $k \neq \bar{\imath}$, which yields the equivalence (3.1.13). For indices $1 \leq p \leq m$, we have $(i, \bar{p}) \in \Gamma_{0,0}$ and $(k, \bar{p}) \in \Gamma^{\mathrm{I}}_{1,0a}$; whilst the indices $M+1 \leq p \leq k$ satisfy $(i, \bar{p}) \in \Gamma^{\mathrm{I}}_{0,1b}$ and $(k, \bar{p}) \in \Gamma_{1,1}$. Thus, we obtain equation (3.1.14) and ultimately deduce $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

When $(i, j) \in \Gamma^{\mathrm{I}}_{1,0}$ where $\widehat{m}+1 \leq j$, we have $1 \leq k \leq m$. Since $(k, \bar{k}) \in \Gamma_{0,0}$, the get the equivalence (3.1.21). For indices $1 \leq p \leq k$, we have both $(i, \bar{p}) \in \Gamma^{\mathrm{I}}_{1,0}$ and $(k, \bar{p}) \in \Gamma_{0,0}$, so we can deduce (3.1.22) and hence $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

Lastly, when $M+n+1 \leq i < j \leq M+N$, we have $M+1 \leq k \leq M+n$. Since $(k, \bar{k}) \in \Gamma_{1,1}$ and $i \neq j = \bar{k}$, we also get the equivalence (3.1.13). For indices $1 \leq p \leq m$, we have $(i, \bar{p}) \in \Gamma^{\mathrm{I}}_{1,0b}$ and $(k, \bar{p}) \in \Gamma^{\mathrm{I}}_{1,0a}$; whilst for the indices $M+1 \leq p \leq k$, both $(i, \bar{p})$ and $(k, \bar{p})$ lie in $\Gamma_{1,1}$. Thus, we obtain (3.1.14) which implies $T_{i\bar{k}}(u)T_{kk}(v) \equiv 0$.

(ii) Again, we shall only provide a proof for $\Theta = \mathrm{I}$ as the case $\Theta = \mathrm{II}$ is similar; accordingly, throughout the proof we shall suppose $(i,j) \in \Lambda_{\mathrm{I}}^{+}$ and $1 \le k \le M+N$.

*Step 1.* Recall that we have the decomposition $(\mathbb{Z}_{M+N}^{+})^2 = \Lambda_{\mathrm{I}}^{+} \cup \Lambda^{\circ} \cup \Lambda_{\mathrm{I}}^{-}$. Assuming $k \neq \widehat{m}$ (i.e., $k \neq \bar{k}$) when $M$ is odd, we observe

$$\big[T_{kk}(u), T_{kk}(v)\big] = \frac{1}{u-v}(-1)^{[k]}\Big(T_{kk}(u)T_{kk}(v) - T_{kk}(v)T_{kk}(u)\Big)$$

and hence, $\big[T_{kk}(u), T_{kk}(v)\big] = 0$. Furthermore, if $(k,l) \in \Lambda_{\mathrm{I}}^{+}$ such that $k \neq \bar{l}$, we have

$$\big[T_{kk}(u), T_{ll}(v)\big] = \frac{1}{u-v}(-1)^{[k]}\Big(T_{lk}(u)T_{kl}(v) - T_{lk}(v)T_{kl}(u)\Big) \equiv 0.$$

Since $(k,l) \in \Lambda_{\mathrm{I}}^{+}$ if and only if $(l,k) \in \Lambda_{\mathrm{I}}^{-}$, and $\big[T_{kk}(u), T_{ll}(v)\big] = -\big[T_{ll}(v), T_{kk}(u)\big]$, all that remains to verify (3.1.12) is to examine when $k = \bar{l}$. To this end, it suffices to show $\big[T_{ll}(u), T_{\bar{l}\bar{l}}(v)\big] \equiv 0$ for $1 \le l \le \widehat{m}$ and $M+1 \le l \le M+n$. Moreover, for the remaining steps we shall define the element

$$\mathbf{A}_{kl} := T_{kl}(u)T_{\bar{k}\bar{l}}(v) - (-1)^{[k]+[l]}T_{\bar{l}\bar{k}}(v)T_{lk}(u). \tag{3.1.23}$$

for any $1 \le k, l \le M+N$.

*Step 2.* First suppose $1 \le l \le m$. Since $\mathbf{A}_{ll} = \big[T_{ll}(u), T_{\bar{l}\bar{l}}(v)\big]$, the defining relations (2.2.8) gives

$$\mathbf{A}_{ll} \equiv -\frac{1}{u-v-\kappa}\sum_{p=1}^{l}\Big(T_{pl}(u)T_{\bar{p}\bar{l}}(v) - T_{\bar{l}\bar{p}}(v)T_{lp}(u)\Big) \equiv -\frac{1}{u-v-\kappa}\sum_{k=1}^{l}\mathbf{A}_{kl}. \tag{3.1.24}$$

Since for such indices, $\mathbf{A}_{kl} \equiv \big[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\big] + \big[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\big]$, we have

$$\mathbf{A}_{kl} \equiv -\frac{1}{u-v-\kappa}\sum_{p=1}^{k}\mathbf{A}_{pk} - \frac{1}{u-v-\kappa}\sum_{p=1}^{l}\mathbf{A}_{pl}. \tag{3.1.25}$$

The equivalences (3.1.24) and (3.1.25) therefore imply the relation $\mathbf{A}_{kl} \equiv \mathbf{A}_{kk} + \mathbf{A}_{ll}$ for indices $1 \le k < l \le m$. Using this resulting relation along with (3.1.24), we derive the formula

$$\left(1 + \frac{l}{u-v-\kappa}\right)\mathbf{A}_{ll} + \frac{1}{u-v-\kappa}\sum_{k=1}^{l-1}\mathbf{A}_{kk} \equiv 0.$$

Hence, an inductive argument will show $\mathbf{A}_{ll} \equiv 0$ for all $1 \leq l \leq m$.

*Step 3.* Let us now suppose $M+1 \leq l \leq M+n$. Since $\mathbf{A}_{ll} = \big[T_{ll}(u), T_{\bar{l}\bar{l}}(v)\big]$, the defining relations (2.2.8) imply

$$\mathbf{A}_{ll} \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1\leq k\leq m,\\ M+1\leq k\leq l}} (-1)^{[k][l]} \mathbf{A}_{kl}. \tag{3.1.26}$$

For indices $M+1 \leq k < l \leq M+n$, $\mathbf{A}_{kl} \equiv \big[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\big] + \big[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\big]$, so

$$\mathbf{A}_{kl} \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1\leq p\leq m,\\ M+1\leq p\leq k}} (-1)^{[p][k]} \mathbf{A}_{pk} - \frac{1}{u-v-\kappa} \sum_{\substack{1\leq p\leq m,\\ M+1\leq p\leq l}} (-1)^{[p][l]} \mathbf{A}_{pl}. \tag{3.1.27}$$

The equivalences (3.1.26) and (3.1.27) imply $\mathbf{A}_{kl} \equiv \mathbf{A}_{kk} + \mathbf{A}_{ll}$ for $M+1 \leq k < l \leq M+n$. For indices $1 \leq k \leq m$ and $M+1 \leq l \leq M+n$, $\mathbf{A}_{kl} \equiv \big[T_{kl}(u), T_{\bar{k}\bar{l}}(v)\big] + \big[T_{lk}(u), T_{\bar{l}\bar{k}}(v)\big]$, so

$$\mathbf{A}_{kl} \equiv \frac{1}{u-v-\kappa} \sum_{1\leq p\leq k} \mathbf{A}_{pk} - \frac{1}{u-v-\kappa} \sum_{\substack{1\leq p\leq m,\\ M+1\leq p\leq l}} (-1)^{[p][l]} \mathbf{A}_{pl}. \tag{3.1.28}$$

The equivalences (3.1.24), (3.1.26), and (3.1.28) therefore imply $\mathbf{A}_{kl} \equiv \mathbf{A}_{ll} - \mathbf{A}_{kk}$ for indices $1 \leq k \leq m$ and $M+1 \leq l \leq M+n$. By combining this new relation with (3.1.26), we can deduce the formula

$$\left(1 + \frac{m+M-l}{u-v-\kappa}\right) \mathbf{A}_{ll} - \frac{1}{u-v-\kappa} \sum_{k=M+1}^{l-1} \mathbf{A}_{kk} \equiv 0$$

since $\mathbf{A}_{kk} \equiv 0$ for $1 \leq k \leq m$ by Step 2. Hence, an inductive argument will prove $\mathbf{A}_{ll} \equiv 0$ for $M+1 \leq l \leq M+n$.

*Step 4.* In the special case when $l = \widehat{m}$ and $M$ is odd, $\mathbf{A}_{ll} = \big[T_{ll}(u), T_{ll}(v)\big]$, so

$$\mathbf{A}_{ll} \equiv \frac{1}{u-v} \mathbf{A}_{ll} - \frac{1}{u-v-\kappa} \sum_{\substack{1\leq k\leq l,\\ M+1\leq k\leq M+n}} (-1)^{[k][l]} \mathbf{A}_{kl} \tag{3.1.29}$$

For indices $1 \leq k < l = \widehat{m}$, $\mathbf{A}_{kl} \equiv \big[T_{kl}(u), T_{\bar{k}l}(v)\big] + \big[T_{lk}(u), T_{l\bar{k}}(v)\big]$, so we have

$$\mathbf{A}_{kl} \equiv -\frac{1}{u-v-\kappa} \sum_{1\leq p\leq k} \mathbf{A}_{pk} - \frac{1}{u-v-\kappa} \sum_{\substack{1\leq p\leq l,\\ M+1\leq p\leq M+n}} (-1)^{[p][l]} \mathbf{A}_{pl}. \tag{3.1.30}$$

Hence, the equivalences (3.1.24), (3.1.29), and (3.1.30) imply $\mathbf{A}_{kl} \equiv \mathbf{A}_{kk} + \frac{u-v-1}{u-v}\mathbf{A}_{ll}$ for $1 \le k < l = \widehat{m}$. Furthermore, since $\mathbf{A}_{kl} \equiv \left[T_{kl}(u), T_{\bar{k}l}(v)\right] + \left[T_{lk}(u), T_{l\bar{k}}(v)\right]$ for indices $M+1 \le k \le M+n$, we have

$$\mathbf{A}_{kl} \equiv -\frac{1}{u-v-\kappa} \sum_{\substack{1 \le p \le m, \\ M+1 \le p \le k}} \mathbf{A}_{pk} + \frac{1}{u-v-\kappa} \sum_{\substack{1 \le p \le l, \\ M+1 \le p \le M+n}} (-1)^{[p][l]} \mathbf{A}_{pl}. \tag{3.1.31}$$

Thus, the equivalences (3.1.26), (3.1.29), and (3.1.31) imply $\mathbf{A}_{kl} \equiv \mathbf{A}_{kk} + \frac{1-(u-v)}{u-v}\mathbf{A}_{ll}$ for $M+1 \le k \le M+n$. Combining these new relations with (3.1.29) will yield

$$\left(1 - \frac{1}{u-v} + \frac{1}{u-v-\kappa} + \frac{(l-1-n)(u-v-1)}{(u-v)(u-v-\kappa)}\right) \mathbf{A}_{ll} \equiv 0$$

since $\mathbf{A}_{kk} \equiv 0$ for $1 \le k \le m$ and $M+1 \le k \le M+n$ by Steps 2 and 3. □

Leveraging the lemma we have just proved, we can now address the first theorem of section §3.1.

**Theorem 3.1.3.** *Let* $\Theta$ *denote either* I *or* II. *Every finite-dimensional irreducible representation* $V$ *of the extended Yangian* $\mathrm{X}(\mathfrak{osp}_{M|N})$ *is an* $X_\Theta$*-highest weight representation. The* $X_\Theta$*-highest weight vector of* $V$ *is unique up to scalar multiple.*

*Proof.* Let $V$ denote a finite-dimensional irreducible representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ and define the subspace

$$V^0 := \left\{ v \in V \mid T_{ij}(u)v = 0 \ \text{ for all } \ (i,j) \in \Lambda_\Theta^+ \right\}. \tag{3.1.32}$$

We claim $V^0$ is non-trivial. Regarding $V$ as an $\mathfrak{osp}_{M|N}$-module under the embedding (2.4.7), there is a partial ordering '$\preccurlyeq$' on its set of weights by stipulating that for any weights $\alpha, \beta \in \mathfrak{h}^*$, one has $\alpha \preccurlyeq \beta$ if and only if $\beta - \alpha$ is an $\mathbb{N}$-linear combination of positive roots in $\Phi_\Theta^+$.

Since the set $\{F_{hh} \mid 1 \le h \le \lfloor \frac{M}{2} \rfloor,\ M+1 \le h \le M+\frac{N}{2}\}$ consists of pairwise commuting elements, their actions on $V$ form a family of pairwise commuting operators, implying that these operators must share a simultaneous eigenvector as $\dim V < \infty$. Hence, since the set of $\mathfrak{osp}_{M|N}$-weights is non-empty and finite, then $V$ must have a maximal weight $\mu$ with respect to the partial ordering '$\preccurlyeq$'.

Letting $v$ be a weight vector corresponding to $\mu$, the assertion follows if $v \in V^0$, so we may assume $v \notin V^0$ and therefore $T_{ij}^{(n)}v \neq 0$ for some $(i,j) \in \Lambda_{\Theta}^{+}$ and $n \in \mathbb{Z}^+$. However, since

$$F_{hh}T_{ij}^{(n)}v = T_{ij}^{(n)}F_{hh}v + \big[F_{hh}, T_{ij}^{(n)}\big]v,$$

we conclude from equation (3.1.4) that the weight of $T_{ij}^{(n)}v$ is of the form $\mu+\alpha$ for some positive root $\alpha \in \Phi_{\Theta}^{+}$, contradicting the maximality of $\mu$ and proving the claim.

By Lemma 3.1.2, the actions of the generators $\{T_{kk}^{(n)} \mid 1 \leq k \leq M+N,\ n \in \mathbb{Z}^+\}$ form a family of pairwise commuting operators on $V^0$. As $V^0$ is a non-trivial subspace of $V$, there must exist a simultaneous eigenvector $0 \neq \xi \in V^0$ for such operators: $T_{kk}^{(n)}\xi = \lambda_k^{(n)}\xi$ for complex eigenvalues $\lambda_k^{(n)}$, $1 \leq k \leq M+N$, $n \in \mathbb{Z}^+$. Via the irreducibility of $V$, we conclude $\mathrm{X}(\mathfrak{osp}_{M|N})\xi = V$, and by collecting these eigenvalues into power series $\lambda_k(u) = 1 + \sum_{n=1}^{\infty}\lambda_k^{(n)}u^{-n}$ we observe the vector $\xi$ satisfies the conditions (3.1.9), so $V$ is a highest weight representation with highest weight vector $\xi$ and highest weight $(\lambda_k(u))_{k=1}^{M+N}$.

It remains to show that $\xi$ is unique up to scalar multiplication. Recalling the PBW Theorem 2.4.5 for $\mathrm{X}(\mathfrak{osp}_{M|N})$, we fix a total order '$\preceq$' on the set $\mathsf{X}$ in such a way that for any $T_{i_1j_1}^{(n_1)}, T_{i_2j_2}^{(n_2)}, T_{i_3j_3}^{(n_3)} \in \mathsf{X}$ satisfying $(i_1,j_1) \in \mathcal{B}_{M|N} \cap \Lambda_{\Theta}^{-}$, $(i_2,j_2) \in \mathcal{B}_{M|N} \cap \Lambda^{\circ}$, and $(i_3,j_3) \in \mathcal{B}_{M|N} \cap \Lambda_{\Theta}^{+}$, then $T_{i_1j_1}^{(n_1)} \preceq T_{i_2j_2}^{(n_2)} \preceq T_{i_3j_3}^{(n_3)}$. Since $V$ is irreducible and finite-dimensional, Schur's lemma infers that each generator $\mathcal{Z}_r$ of the center $\mathrm{ZX}(\mathfrak{osp}_{M|N})$ acts by a scalar. Therefore, by the total ordering on $\mathsf{X}$, we conclude that $V$ is spanned by ordered elements of the form

$$T_{i_1j_1}^{(n_1)} \cdots T_{i_kj_k}^{(n_k)}\xi, \tag{3.1.33}$$

where $k \in \mathbb{N}$, $(i_p,j_p) \in \mathcal{B}_{M|N} \cap \Lambda_{\Theta}^{-}$, and $n_p \in \mathbb{Z}^+$ for $1 \leq p \leq k$. Furthermore, since $F_{hh} = (-1)^{[h]}T_{hh}^{(1)} - \frac{1}{2}(-1)^{[h]}\mathcal{Z}_1$, then $\xi$ is also a weight vector of some weight $\mu$. By (3.1.4), the elements (3.1.33) will therefore be weight vectors with corresponding weights of the form $\mu + \sum_{p=1}^{k}\alpha_p$, where $\alpha_p \in \Phi_{\Theta}^{-}$.

Hence, there is a weight space decomposition $V = \bigoplus_{\nu\in\mathfrak{h}^*}V_\nu$ where each weight $\nu \neq \mu$ is of the form $\mu - \sum_{p=1}^{k}\alpha_p$ for $\alpha_p \in \Phi_{\Theta}^{+}$. As a result, the space $V_\mu$ has dimension 1 and is given by $V_\mu = \mathrm{span}_{\mathbb{C}}\{\xi\}$. If $\tilde{\xi}$ is another highest weight vector of $V$ of highest weight $(\lambda_k(u))_{k=1}^{M+N}$, the weight space decomposition ensures that its $\mathfrak{osp}_{M|N}$-weight must be equal to $\mu$. Hence, $\tilde{\xi} \in V_\mu$, showing $\tilde{\xi} = c\xi$ for some $c \in \mathbb{C}^*$. □

As we saw in the proof of Theorem 3.1.3, Schur's lemma infers that central elements in $\mathrm{X}(\mathfrak{osp}_{M|N})$ act on finite-dimensional irreducible representations by scalars. As we show in the following proposition, we can determine these scalars explicitly.

**Proposition 3.1.4.** *Let* $\Theta$ *denote either* I *or* II *and let* $V$ *be an* $X_\Theta$*-highest weight representation of* $\mathrm{X}(\mathfrak{osp}_{M|N})$ *with highest weight* $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ *and highest weight vector* $\xi_\Theta$*. For* $\Theta = \mathrm{I}$*,* $\mathcal{Z}(u)$ *acts on* $\xi_\Theta$ *by* $\lambda_1(u+\kappa)\lambda_M(u)$ *and for* $\Theta = \mathrm{II}$*,* $\mathcal{Z}(u)$ *acts on* $\xi_\Theta$ *by* $\lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)$*.*

*Proof.* Let $\xi_\Theta$ be a highest weight vector of $V$ so that $V = \mathrm{X}(\mathfrak{osp}_{M|N})\xi_\Theta$. If $\Theta = \mathrm{I}$, setting $i = j = M$ in equation (2.2.18) gives $\mathcal{Z}(u) = \sum_{k=1}^{M+N}(-1)^{[k]}T_{\bar{k}1}(u+\kappa)T_{kM}(u)$, so

$$\mathcal{Z}(u)\xi_{\mathrm{I}} = T_{11}(u+\kappa)T_{MM}(u)\xi_{\mathrm{I}} = \lambda_1(u+\kappa)\lambda_M(u)\xi_{\mathrm{I}}.$$

Otherwise when $\Theta = \mathrm{II}$, we may designate $i = j = M+N$ in equation (2.2.18) to provide $\mathcal{Z}(u) = -\sum_{k=1}^{M+N}T_{\bar{k},M+1}(u+\kappa)T_{k,M+N}(u)$, so

$$\mathcal{Z}(u)\xi_{\mathrm{II}} = T_{M+1,M+1}(u+\kappa)T_{M+N,M+N}(u)\xi_{\mathrm{II}} = \lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)\xi_{\mathrm{II}}.$$

□

Furthermore, there are some immediate relations of the components of highest weights:

**Proposition 3.1.5.** *Let* $\Theta$ *denote either* I *or* II *and let* $V$ *be an* $X_\Theta$*-highest weight representation of* $\mathrm{X}(\mathfrak{osp}_{M|N})$ *with highest weight* $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ *and highest weight vector* $\xi_\Theta$*. If* $\Theta = \mathrm{I}$ *and* $M \geq 4$*, then*

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{\lambda_{M-1}(u-\kappa+1)}{\lambda_M(u-\kappa+1)}, \tag{3.1.34}$$

*or if* $\Theta = \mathrm{II}$ *and* $N \geq 4$*, then*

$$\frac{\lambda_{M+1}(u)}{\lambda_{M+2}(u)} = \frac{\lambda_{M+N-1}(u-\kappa-1)}{\lambda_{M+N}(u-\kappa-1)}. \tag{3.1.35}$$

*Proof.* By first assuming $\Theta = \mathrm{I}$ and $M \geq 4$, the defining relations (2.2.8) infer

$$\begin{aligned} T_{12}(u)T_{M,M-1}(v)\xi_{\mathrm{I}} &= \big[T_{12}(u), T_{M,M-1}(v)\big]\xi_{\mathrm{I}} \\ &= -\frac{1}{u-v-\kappa}\Big(T_{12}(u)T_{M,M-1}(v) + \lambda_2(u)\lambda_{M-1}(v) - \lambda_1(u)\lambda_M(v)\Big)\xi_{\mathrm{I}}, \end{aligned}$$

so $(u-v-\kappa+1)T_{12}(u)T_{M,M-1}(v)\xi_{\mathrm{I}} = \big(\lambda_2(u)\lambda_{M-1}(v) - \lambda_1(u)\lambda_M(v)\big)\xi_{\mathrm{I}}$. Evaluating at $v = u-\kappa+1$ then yields the desired relation. Similarly, if instead $\Theta = \mathrm{II}$ and $N \geq 4$, the defining relations (2.2.8) show

$$\begin{aligned}
&T_{M+1,M+2}(u)T_{M+N,M+N-1}(v)\xi_{\mathrm{II}} = \big[T_{M+1,M+2}(u), T_{M+N,M+N-1}(v)\big]\xi_{\mathrm{II}} \\
&= \frac{1}{u-v-\kappa}\Big(T_{M+1,M+2}(u)T_{M+N,M+N-1}(v) + \lambda_{M+2}(u)\lambda_{M+N-1}(v) - \lambda_{M+1}(u)\lambda_{M+N}(v)\Big)\xi_{\mathrm{II}},
\end{aligned}$$

so setting $v = u-\kappa-1$ in the equation

$$\begin{aligned}
(u-v-\kappa-1)&T_{M+1,M+2}(u)T_{M+N,M+N-1}(v)\xi_{\mathrm{II}} \\
&= \big(\lambda_{M+2}(u)\lambda_{M+N-1}(v) - \lambda_{M+1}(u)\lambda_{M+N}(v)\big)\xi_{\mathrm{II}}
\end{aligned}$$

will yield the desired relation. □

### 3.1.3 Restriction functors from $\operatorname{Rep} \mathrm{X}(\mathfrak{osp}_{M|N})$

One can embed the lower rank Lie superalgebras $\mathfrak{osp}_{(M-2)|N}$ and $\mathfrak{osp}_{M|(N-2)}$ within the ambient orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|N}$, thereby allowing one to pullback representations of $\mathfrak{osp}_{M|N}$ to those of $\mathfrak{osp}_{(M-2)|N}$ and $\mathfrak{osp}_{M|(N-2)}$. On the level of Yangians, the maps $\mathrm{X}(\mathfrak{osp}_{(M-2)|N}) \to \mathrm{X}(\mathfrak{osp}_{M|N})$ and $\mathrm{X}(\mathfrak{osp}_{M|(N-2)}) \to \mathrm{X}(\mathfrak{osp}_{M|N})$ that imitate these Lie superalgebra inclusions are, however, *not* superalgebra morphisms. The purpose of this section is to find an alternate construction to solve this problem.

First, let us consider the following subsets of $(\mathbb{Z}^+)^2$:

$$\begin{aligned}
\mathsf{M}_{0,0} = \big\{(i,j) \mid 2 \leq i,j \leq M-1\big\}, \quad \mathsf{M}_{1,1} &= \big\{(i,j) \mid M+1 \leq i,j \leq M+N\big\}, \\
\mathsf{M}_{0,1} &= \big\{(i,j) \mid 2 \leq i \leq M-1,\ M+1 \leq j \leq M+N\big\}, \\
\mathsf{M}_{1,0} &= \big\{(i,j) \mid M+1 \leq i \leq M+N,\ 2 \leq j \leq M-1\big\},
\end{aligned}$$

$$\begin{aligned}
\mathsf{N}_{0,0} = \big\{(i,j) \mid 1 \leq i,j \leq M\big\}, \quad \mathsf{N}_{1,1} &= \big\{(i,j) \mid M+2 \leq i,j \leq M+N-1\big\}, \\
\mathsf{N}_{0,1} &= \big\{(i,j) \mid 1 \leq i \leq M,\ M+2 \leq j \leq M+N-1\big\}, \\
\mathsf{N}_{1,0} &= \big\{(i,j) \mid M+2 \leq i \leq M+N-1,\ 1 \leq j \leq M\big\},
\end{aligned}$$

so we can define

$$\mathsf{M} := \mathsf{M}_{0,0} \cup \mathsf{M}_{0,1} \cup \mathsf{M}_{1,0} \cup \mathsf{M}_{1,1} \quad \text{and} \quad \mathsf{N} := \mathsf{N}_{0,0} \cup \mathsf{N}_{0,1} \cup \mathsf{N}_{1,0} \cup \mathsf{N}_{1,1}. \tag{3.1.36}$$

Setting

$$\mathfrak{m} = \mathrm{span}_{\mathbb{C}}\{F_{ij}\}_{(i,j)\in \mathsf{M}} \quad \text{and} \quad \mathfrak{n} = \mathrm{span}_{\mathbb{C}}\{F_{ij}\}_{(i,j)\in \mathsf{N}},$$

these are Lie sub-superalgebras of $\mathfrak{osp}_{M|N}$ of ranks $m+n-1$ which represent the embeddings of $\mathfrak{osp}_{(M-2)|N}$ and $\mathfrak{osp}_{M|(N-2)}$ within $\mathfrak{osp}_{M|N}$, respectively:

$$\mathfrak{osp}_{(M-2)|N} \xrightarrow{\sim} \mathfrak{m} \hookrightarrow \mathfrak{osp}_{M|N} \quad \text{and} \quad \mathfrak{osp}_{M|(N-2)} \xrightarrow{\sim} \mathfrak{n} \hookrightarrow \mathfrak{osp}_{M|N}.$$

We now consider the following proposition:

**Proposition 3.1.6.** (i) *Let $\mathcal{I}^+$ be the left graded ideal of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{1k}(u)$ and $T_{lM}(u)$ for indices $2 \le k \le M$, $1 \le l \le M-1$, and $M+1 \le k,l \le M+N$. There is a superalgebra morphism*

$$\mathrm{X}(\mathfrak{osp}_{(M-2)|N}) \to \mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+, \quad \acute{T}_{ij}(u) \mapsto T_{\nu(i)\nu(j)}(u) \mod \mathcal{I}^+,$$

*where $\acute{T}_{ij}(u)$ denotes a generating series for $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})$ and $\nu(i) = i+1$ for $1 \le i \le M-2$, whereas $\nu(i) = i+2$ for $M-1 \le i \le M+N-2$.*

(ii) *Let $\mathcal{I}_+$ be the left graded ideal of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{M+1,k}(u)$ and $T_{l,M+N}(u)$ for indices $1 \le k,l \le M$, $M+2 \le k \le M+N$, and $M+1 \le l \le M+N-1$. There is a superalgebra morphism*

$$\mathrm{X}(\mathfrak{osp}_{M|(N-2)}) \to \mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}_+, \quad \grave{T}_{ij}(u) \mapsto T_{\nu'(i)\nu'(j)}(u) \mod \mathcal{I}_+,$$

*where $\grave{T}_{ij}(u)$ denotes a generating series for $\mathrm{X}(\mathfrak{osp}_{M|(N-2)})$ and $\nu'(i) = i$ for $1 \le i \le M$, whereas $\nu'(i) = i+1$ for $M+1 \le i \le M+N-2$.*

We observe that the maps

$$\begin{aligned} \mathrm{X}(\mathfrak{osp}_{(M-2)|N}) \to \mathrm{X}(\mathfrak{osp}_{M|N}), \quad & \acute{T}_{ij}(u) \mapsto T_{\nu(i)\nu(j)}(u) \\ \text{and} \quad \mathrm{X}(\mathfrak{osp}_{M|(N-2)}) \to \mathrm{X}(\mathfrak{osp}_{M|N}), \quad & \grave{T}_{ij}(u) \mapsto T_{\nu'(i)\nu'(j)}(u) \end{aligned}$$

will not be superalgebra morphisms, thereby instigating one to descend to a certain quotient of $\mathrm{X}(\mathfrak{osp}_{M|N})$ as in the above proposition. We note, however, that a true embedding of $\mathrm{X}(\mathfrak{osp}_{M|(N-2)})$ in $\mathrm{X}(\mathfrak{osp}_{M|N})$ has recently been established in the paper [Mol23a, §3] via the use of *quasideterminants*. Moreover, such embedding is compatible with Proposition 3.1.6. For our purposes, the above morphisms are sufficient in regards to studying the representation theory of the extended Yangians in this work (c.f. Proposition 3.1.9), so we continue with the proof of the above result.

*Proof of Proposition 3.1.6.* We shall provide a proof for part (i) as (ii) is similar. Accordingly, we shall suppose $(i,j),(k,l) \in \mathsf{M}$ for the duration of the proof and shall use '$\equiv$' to denote equivalence of elements in $X(\mathfrak{osp}_{M|N})$ modulo $\mathcal{I}^+$ for brevity. By the defining relations (2.2.8), we have

$$\begin{aligned}\big[T_{ij}(u), T_{kl}(v)\big] \equiv{}& \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\Big)\\ &- \frac{1}{u-v-\kappa}\Bigg(\delta_{\bar{\imath}k}\sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v)\\ &\qquad - \delta_{\bar{\jmath}l}\sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg)\\ &- \frac{1}{u-v-\kappa}\Big(\delta_{\bar{\imath}k}(-1)^{[i][j]+[i]}\theta_i T_{1j}(u)T_{Ml}(v)\\ &\qquad - \delta_{\bar{\jmath}l}(-1)^{[i][k]+[j][k]+[j]}\theta_j T_{kM}(v)T_{i1}(u)\Big).\end{aligned}$$

Via the defining relations again along with $T_{1j}(u)T_{Ml}(v) \equiv \big[T_{1j}(u), T_{Ml}(v)\big]$, we deduce

$$\begin{aligned}T_{1j}(u)T_{Ml}(v) \equiv{}& -\frac{1}{u-v-\kappa+1}\sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}}(-1)^{[j][p]}\theta_p T_{pj}(u)T_{\bar{p}l}(v)\\ &+ \frac{1}{u-v-\kappa+1}\delta_{\bar{\jmath}l}(-1)^{[j]}\theta_j T_{MM}(v)T_{11}(u).\end{aligned}$$

Analogously, we compute

$$\begin{aligned}T_{kM}(v)T_{i1}(u) &\equiv -(-1)^{([i]+[1])([k]+[M])}\big[T_{i1}(u), T_{kM}(v)\big]\\ &\equiv \frac{1}{u-v-\kappa}\Big(\delta_{\bar{\imath}k}\theta_i T_{11}(u)T_{MM}(v) - T_{kM}(v)T_{i1}(u)\Big)\\ &\qquad - \frac{1}{u-v-\kappa}\sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}}(-1)^{[i][p]+[p]}\theta_p T_{k\bar{p}}(v)T_{ip}(u)\end{aligned}$$

and hence

$$\begin{aligned}T_{kM}(v)T_{i1}(u) \equiv{}& \frac{1}{u-v-\kappa+1}\delta_{\bar{\imath}k}\theta_i T_{11}(u)T_{MM}(v)\\ &- \frac{1}{u-v-\kappa+1}\sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}}(-1)^{[i][p]+[p]}\theta_p T_{k\bar{p}}(v)T_{ip}(u).\end{aligned}$$

Combining everything, we obtain

$$\begin{aligned}\big[T_{ij}(u),T_{kl}(v)\big] \equiv{}& \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\Big(T_{kj}(u)T_{il}(v)-T_{kj}(v)T_{il}(u)\Big)\\ &-\frac{1}{u-v-\kappa+1}\Bigg(\delta_{\bar{\imath}k}\sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}}(-1)^{[i][j]+[i]+[j][p]}\theta_i\theta_p T_{pj}(u)T_{\bar{p}l}(v)\\ &\qquad-\delta_{\bar{\jmath}l}\sum_{\substack{2\le p\le M-1,\\ M+1\le p\le M+N}}(-1)^{[i][k]+[j][k]+[j]+[i][p]+[p]}\theta_j\theta_p T_{k\bar{p}}(v)T_{ip}(u)\Bigg)\\ &+\frac{1}{(u-v-\kappa)(u-v-\kappa+1)}\delta_{\bar{\imath}k}\delta_{\bar{\jmath}l}(-1)^{[i][j]+[i]+[j]}\theta_i\theta_j\big[T_{11}(u),T_{MM}(v)\big].\end{aligned}$$

Lastly, the relations (2.2.8) imply

$$\big[T_{11}(u),T_{MM}(v)\big]\equiv-\frac{1}{u-v-\kappa}\big[T_{11}(u),T_{MM}(v)\big],$$

meaning $\big[T_{11}(u),T_{MM}(v)\big]\equiv 0$ and therefore the desired relations are satisfied for the operators $T_{ij}(u)$, $(i,j)\in\mathsf{M}$, on $V^+$, since $\kappa_{M-2,N}=\kappa_{M,N}-1$ is the parameter associated to the Lie superalgebra $\mathfrak{m}\cong\mathfrak{osp}_{(M-2)|N}$. □

If $\iota_M$ denotes the superalgebra morphism $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})\to\mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}_M^+$ in the above proposition, where we write $\mathcal{I}_M^+$ for the left graded ideal $\mathcal{I}^+$, then there is an induced superalgebra morphism

$$\widetilde{\iota_M}\colon\ \mathrm{X}(\mathfrak{osp}_{(M-2)|N})/\mathcal{I}_{M-2}^+\to\mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^{\Sigma_2+},$$

where $\mathcal{I}^{\Sigma_2+}$ is the left graded ideal generated by the coefficients of $T_{ik}(u)$ and $T_{li}(u)$ for indices $i=1,2$; $i+1\le k\le M$, $1\le l\le M-i$, and $M+1\le k,l\le M+N$. Consequently, the composition $\widetilde{\iota_M}\circ\iota_{M-2}$ describes a superalgebra morphism from $\mathrm{X}(\mathfrak{osp}_{(M-4)|N})$ to $\mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^{\Sigma_2+}$.

Depending on the parity of $M$, we can therefore construct a superalgebra morphism from either $\mathrm{X}(\mathfrak{osp}_{0|N})$ or $\mathrm{X}(\mathfrak{osp}_{1|N})$ to $\mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^{\Sigma_m+}$ for a certain left graded ideal $\mathcal{I}^{\Sigma_m+}$. Accordingly, one can similarly construct a superalgebra morphism from $\mathrm{X}(\mathfrak{osp}_{M|0})$ to $\mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}_{\Sigma_n+}$ for a particular left graded ideal $\mathcal{I}_{\Sigma_n+}$. We summarize these observations in the following remark.

**Remark 3.1.7.** (i) Let $m = \lfloor \frac{M}{2} \rfloor$ and define $\mathcal{I}^{\Sigma_m+}$ to be the left graded ideal of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{ij}(u)$ for indices $1 \leq i < j \leq M$; $1 \leq i \leq m$, $M+1 \leq j \leq M+N$; and $M+1 \leq i \leq M+N$, $\widehat{m}+1 \leq j \leq M$. There is a superalgebra morphism

$$\mathrm{X}(\mathfrak{osp}_{(M-2m)|N}) \to \mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^{\Sigma_m+}, \quad \acute{T}_{ij}(u) \mapsto T_{\nu(i)\nu(j)}(u) \mod \mathcal{I}^{\Sigma_m+},$$

where $\acute{T}_{ij}(u)$ denotes a generating series for $\mathrm{X}(\mathfrak{osp}_{(M-2m)|N})$ and $\nu(i) = i + m$ for $1 \leq i \leq M-2m$, whereas $\nu(i) = 2m + i$ for $M-2m+1 \leq i \leq M-2m+N$.

(ii) Let $n = \frac{N}{2}$ and define $\mathcal{I}_{\Sigma_n+}$ to be the left graded ideal of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{ij}(u)$ for indices $1 \leq i \leq M$, $M+n+1 \leq j \leq M+N$; $M+1 \leq i \leq M+n$, $1 \leq j \leq M$, and $M+1 \leq i < j \leq M+N$. There is a superalgebra morphism

$$\mathrm{X}(\mathfrak{osp}_{M|0}) \to \mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}_{\Sigma_n+}, \quad \grave{T}_{ij}(u) \mapsto T_{\nu'(i)\nu'(j)}(u) \mod \mathcal{I}_{\Sigma_n+},$$

where $\grave{T}_{ij}(u)$ denotes a generating series for $\mathrm{X}(\mathfrak{osp}_{M|0})$ and $\nu'(i) = i$ for indices $1 \leq i \leq M$.

Before proving the main result of this section, we need to establish some relations occurring in $\mathrm{X}(\mathfrak{osp}_{M|N})$ modulo the left graded ideals $\mathcal{I}^+$ or $\mathcal{I}_+$ as in Proposition 3.1.6.

**Lemma 3.1.8.** *Let $\mathcal{I}^+$ and $\mathcal{I}_+$ be the left graded ideals of $\mathrm{X}(\mathfrak{osp}_{M|N})$ as defined in Proposition 3.1.6.*

(i) $T_{1k}(v)T_{ij}(u) \equiv T_{lM}(v)T_{ij}(u) \equiv 0 \mod \mathcal{I}^+$ *for indices* $(i,j) \in \mathsf{M}$, $2 \leq k \leq M$, $1 \leq l \leq M-1$, *and* $M+1 \leq k,l \leq M+N$.

(ii) $T_{M+1,k}(v)T_{ij}(u) \equiv T_{l,M+N}(v)T_{ij}(u) \equiv 0 \mod \mathcal{I}_+$ *for* $(i,j) \in \mathsf{N}$, $1 \leq k,l \leq M$, $M+2 \leq k \leq M+N$, *and* $M+1 \leq l \leq M+N-1$.

*Proof.* We shall provide the proof for (i) as (ii) is similar. Accordingly, we shall suppose $(i,j),(k,l) \in \mathsf{M}$ for the duration of the proof and shall use '$\equiv$' to denote equivalence of elements in $\mathrm{X}(\mathfrak{osp}_{M|N})$ modulo $\mathcal{I}^+$ for brevity.

First supposing $2 \leq k \leq M$ and $(i,j) \in \mathsf{M}_{0,1} \cup \mathsf{M}_{1,1}$, or $M+1 \leq k \leq M+N$ and $(i,j) \in \mathsf{M}_{0,0} \cup \mathsf{M}_{1,0}$, relations (2.2.8) imply $T_{1k}(u)T_{ij}(v) \equiv \big[T_{1k}(u), T_{ij}(v)\big] \equiv 0$.

Alternatively, when $2 \le k \le M$ and $(i,j) \in \mathsf{M}_{0,0} \cup \mathsf{M}_{1,0}$, or $M+1 \le k \le M+N$ and $(i,j) \in \mathsf{M}_{0,1} \cup \mathsf{M}_{1,1}$, the same relations yield

$$T_{1k}(u)T_{ij}(v) \equiv \big[T_{1k}(u), T_{ij}(v)\big] \equiv \frac{\delta_{\bar{k}j}}{u-v-\kappa}(-1)^{[i][k]+[k]}\theta_k T_{iM}(v)T_{11}(u).$$

Since $T_{iM}(v)T_{11}(u) \equiv -\big[T_{11}(u), T_{iM}(v)\big] \equiv -(u-v-\kappa)^{-1}T_{iM}(v)T_{11}(u)$, it follows that $T_{iM}(v)T_{11}(u) \equiv 0$ and hence $T_{1k}(u)T_{ij}(v)\eta \equiv 0$.

Lastly, when $1 \le l \le M-1$ and $(i,j) \in \mathsf{M}_{1,0} \cup \mathsf{M}_{1,1}$, or $M+1 \le l \le M+N$ and $(i,j) \in \mathsf{M}_{0,0} \cup \mathsf{M}_{0,1}$, relations (2.2.8) provide $T_{lM}(v)T_{ij}(u) \equiv -\big[T_{ij}(u), T_{lM}(v)\big] \equiv 0$. Otherwise, if $1 \le l \le M-1$ and $(i,j) \in \mathsf{M}_{0,0} \cup \mathsf{M}_{0,1}$, or $M+1 \le l \le M+N$ and $(i,j) \in \mathsf{M}_{1,0} \cup \mathsf{M}_{1,1}$, the same relations give

$$T_{lM}(v)T_{ij}(u) \equiv -\big[T_{ij}(u), T_{lM}(v)\big] \equiv \frac{\delta_{\bar{i}k}}{u-v-\kappa}(-1)^{[i][j]}\theta_i T_{1j}(u)T_{MM}(v)\eta.$$

As $T_{1j}(u)T_{MM}(v) \equiv \big[T_{1j}(u), T_{MM}(v)\big] \equiv -(u-v-\kappa)^{-1}T_{1j}(u)T_{MM}(v)$, it follows that $T_{1j}(u)T_{MM}(v) \equiv 0$ and hence $T_{lM}(v)T_{ij}(u)\eta \equiv 0$, proving the lemma. □

Any representation $V$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$ will have two important subspaces denoted $V^+$ and $V_+$. To introduce these, we first consider the subspaces

$$\begin{aligned}
V_1 &= \big\{\eta \in V \mid T_{1k}(u)\eta = 0 \ \text{ for } \ 2 \le k \le M \ \text{ and } \ M+1 \le k \le M+N\big\},\\
V_M &= \big\{\eta \in V \mid T_{kM}(u)\eta = 0 \ \text{ for } \ 1 \le k \le M-1 \ \text{ and } \ M+1 \le k \le M+N\big\},\\
V_{M+1} &= \big\{\eta \in V \mid T_{M+1,k}(u)\eta = 0 \ \text{ for } \ 1 \le k \le M \ \text{ and } \ M+2 \le k \le M+N\big\},\\
V_{M+N} &= \big\{\eta \in V \mid T_{k,M+N}(u)\eta = 0 \ \text{ for } \ 1 \le k \le M \ \text{ and } \ M+1 \le k \le M+N-1\big\},
\end{aligned}$$

so that we can accordingly define

$$V^+ := V_1 \cap V_M \quad \text{and} \quad V_+ := V_{M+1} \cap V_{M+N}. \tag{3.1.37}$$

Note that these intersections may be trivial; however, if $V$ is an $X_{\mathrm{I}}$-highest weight representation, then $V^+$ contains the $X_{\mathrm{I}}$-highest weight vector and if $V$ is an $X_{\mathrm{II}}$-highest weight representation, then $V_+$ contains the $X_{\mathrm{II}}$-highest weight vector. In particular, if $V$ is finite-dimensional and irreducible, then Theorem 3.1.3 ensures that $V^+$ and $V_+$ will always be non-trivial.

For a superalgebra $\mathcal{A}$, we shall let $\mathrm{Rep}(\mathcal{A})$ denote its category of representations. We now arrive at the main proposition for this subsection.

**Proposition 3.1.9.** *There are covariant functors*

$$\begin{aligned} \mathcal{F}^+\colon \mathrm{Rep}(\mathrm{X}(\mathfrak{osp}_{M|N})) &\to \mathrm{Rep}(\mathrm{X}(\mathfrak{osp}_{(M-2)|N})), \quad V \mapsto V^+, \quad \phi \mapsto \phi|_{V^+} \\ \text{and} \quad \mathcal{F}_+\colon \mathrm{Rep}(\mathrm{X}(\mathfrak{osp}_{M|N})) &\to \mathrm{Rep}(\mathrm{X}(\mathfrak{osp}_{M|(N-2)})), \quad V \mapsto V_+, \quad \phi \mapsto \phi|_{V_+}, \end{aligned}$$

*where $\mathcal{F}^+(V) = V^+$ and $\mathcal{F}_+(V) = V_+$ are defined by (3.1.37) for any $\mathrm{X}(\mathfrak{osp}_{M|N})$-module $V$ and $\mathcal{F}^+(\phi) = \phi|_{V^+}$ and $\mathcal{F}_+(\phi) = \phi|_{V_+}$ for any $\mathrm{X}(\mathfrak{osp}_{M|N})$-module morphism $\phi$.*

*Proof.* We shall only prove the existence of $\mathcal{F}^+$, as the proof for $\mathcal{F}_+$ is similar. Given a representation $\varphi\colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathrm{End}\, V$, we know by the definition of the left graded ideal $\mathcal{I}^+$ that $\varphi(\mathcal{I}^+)(V^+) = 0$; hence, there is a well-defined action $\overline{\varphi}$ of the quotient $\mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+$ on $V^+$ and composing such with the superalgebra morphism in Proposition 3.1.6 gives

$$\mathrm{X}(\mathfrak{osp}_{(M-2)|N}) \xrightarrow{\iota_M} \mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+ \xrightarrow{\overline{\varphi}} \mathrm{Hom}(V^+, V).$$

We observe that it is not evident $V^+$ should be closed under the action of $\mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}^+$. However, Lemma 3.1.8 ensures that the image of $\overline{\varphi} \circ \iota_M$ lies in $V^+$, so we nonetheless attain a representation $\overline{\varphi} \circ \iota_M\colon \mathrm{X}(\mathfrak{osp}_{(M-2)|N}) \to \mathrm{End}\, V^+$.

Given an $\mathrm{X}(\mathfrak{osp}_{M|N})$-module morphism $\phi\colon V \to W$, the $\mathrm{X}(\mathfrak{osp}_{M|N})$-linearity of $\phi$ implies $\phi|_{V^+}(V^+) \subseteq W^+$. A similar discussion to the above also shows that $\phi|_{V^+}$ is $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})$-linear. □

**Remark 3.1.10.** We observe that if $V$ is an $X_{\mathrm{I}}$-highest weight representation with $X_{\mathrm{I}}$-highest weight vector $\xi$ and $X_{\mathrm{I}}$-highest weight $\big(\lambda_k(u)\big)_{k=1}^{M+N}$, then $\xi \in V^+$ and the $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})$-submodule generated by $\xi$ will be an $X_{\mathrm{I}}$-highest weight representation with $X_{\mathrm{I}}$-highest weight vector $\xi$ and $X_{\mathrm{I}}$-highest weight

$$\big(\lambda_2(u), \ldots, \lambda_{M-1}(u), \lambda_{M+1}(u), \ldots, \lambda_{M+N}(u)\big).$$

Similarly, if $V$ is an $X_{\mathrm{II}}$-highest weight representation with $X_{\mathrm{II}}$-highest weight vector $\xi$ and $X_{\mathrm{II}}$-highest weight $\big(\lambda_k(u)\big)_{k=1}^{M+N}$, then $\xi \in V_+$ and the $\mathrm{X}(\mathfrak{osp}_{M|(N-2)})$-submodule generated by $\xi$ will be an $X_{\mathrm{II}}$-highest weight representation with $X_{\mathrm{II}}$-highest weight vector $\xi$ and $X_{\mathrm{II}}$-highest weight

$$\big(\lambda_1(u),\dots,\lambda_M(u),\lambda_{M+2}(u),\dots,\lambda_{M+N-1}(u)\big).$$

Allowing $\mathcal{F}^+_M$ to denote the restriction functor $\mathcal{F}^+$ in Proposition 3.1.9, one can consider the composition $(\mathcal{F}^+_{M-2}\circ\mathcal{F}^+_M)(V)=(V^+)^+$ for any $\mathrm{X}(\mathfrak{osp}_{M|N})$-module $V$. Via the embedding in Proposition 3.1.6, we observe $(V^+)^+$ is the subspace of $V^+$ consisting of all vectors $\eta\in V^+$ satisfying $T_{k,M-1}(u)\eta=T_{2l}(u)\eta=0$ for indices $2\le k\le M-2$, $3\le l\le M-1$, and $M+1\le k,l\le M+N$. In particular, we note the following remark.

**Remark 3.1.11.** Let $m=\lfloor\frac{M}{2}\rfloor$, $\widehat{m}=\lceil\frac{M}{2}\rceil$, $n=\frac{N}{2}$ and define the subspaces

$$\begin{aligned} V^{\Sigma_m+} &:= \big\{\eta\in V \mid T_{ij}(u)\eta=0 \ \text{ for } \ (i,j)\in\Lambda^+_{\mathrm{I}}\setminus\big(\Gamma_{1,1}\cup\{(k,\widehat{m}),(m+1,\bar{k})\}_{k=M+1}^{M+n}\big)\big\}\\ \text{and}\quad V_{\Sigma_n+} &:= \big\{\eta\in V \mid T_{ij}(u)\eta=0 \ \text{ for } \ (i,j)\in\Lambda^+_{\mathrm{II}}\setminus\Gamma_{1,1}\big\}. \end{aligned} \tag{3.1.38}$$

If $\mathcal{F}^+_M$, $\mathcal{F}^N_+$ denotes the respective restriction functors $\mathcal{F}^+$, $\mathcal{F}_+$ in Proposition 3.1.9, one computes

$$\begin{aligned} (\mathcal{F}^+_{M-2m+2}\circ\cdots\circ\mathcal{F}^+_{M-2}\circ\mathcal{F}^+_M)(V) &= V^{\Sigma_m+}\\ \text{and}\qquad (\mathcal{F}^2_+\circ\cdots\circ\mathcal{F}^{N-2}_+\circ\mathcal{F}^N_+)(V) &= V_{\Sigma_n+}. \end{aligned}$$

In particular, $V^{\Sigma_m+}$ becomes a representation of $\mathrm{X}(\mathfrak{osp}_{(M-2m)|N})$, while $V_{\Sigma_n+}$ becomes a representation of $\mathrm{X}(\mathfrak{osp}_{M|0})\cong\mathrm{X}(\mathfrak{so}_M)$.

### 3.1.4 Verma modules

An essential component of our highest weight theory is the notion of a *Verma module*. In contrast to traditional Lie theory, however, Verma modules here can be trivial.

**Definition 3.1.12.** Let $\Theta$ denote either I or II. Given a tuple $\lambda(u)=(\lambda_k(u))_{k=1}^{M+N}$ of the form (3.1.10) we define the $X_\Theta$-*Verma module* $M_\Theta(\lambda(u))$ to be the quotient:

$$M_\Theta(\lambda(u)) := \mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{I}_\Theta(\lambda(u)), \tag{3.1.39}$$

where $\mathcal{I}_\Theta(\lambda(u))$ is the left graded ideal of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of the series $T_{ij}(u)$ for $(i,j)\in\Lambda^+_\Theta$ and $T_{kk}(u)-\lambda_k(u)\mathbf{1}$ for $1\le k\le M+N$.

When $M_\Theta(\lambda(u))$ is non-trivial, it is an $X_\Theta$-highest weight representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ with $X_\Theta$-highest weight $\lambda(u)$ and $X_\Theta$-highest weight vector $\mathbf{1}_{\lambda(u)}$, the image of $\mathbf{1}$ in the canonical projection $\mathrm{X}(\mathfrak{osp}_{M|N}) \to M_\Theta(\lambda(u))$. Furthermore, if $L$ is an $X_\Theta$-highest weight representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ with highest weight $\lambda(u)$ and highest weight vector $\xi$, then, provided $M(\lambda(u))$ is non-trivial, there is a surjective $\mathrm{X}(\mathfrak{osp}_{M|N})$-module morphism $\varphi\colon M_\Theta(\lambda(u)) \to L$ induced by the assignment $\mathbf{1}_{\lambda(u)} \mapsto \xi$; thus, $L \cong M_\Theta(\lambda(u))/\ker\varphi$.

By (3.1.8) , $\bigoplus_{\mu\in\mathfrak{h}^*} M_\Theta(\lambda(u))_\mu$ is invariant under the action of $\mathrm{X}(\mathfrak{osp}_{M|N})$. Therefore, since $\mathbf{1}_{\lambda(u)}$ is contained in $M_\Theta(\lambda(u))_{\lambda^{(1)}} \subset \bigoplus_{\mu\in\mathfrak{h}^*} M_\Theta(\lambda(u))_\mu$, where $\lambda^{(1)} \in \mathfrak{h}^*$ is the linear functional given by $\lambda^{(1)}(F_{kk}) = \lambda_k^{(1)}$, we have the weight space decomposition

$$M_\Theta(\lambda(u)) = \bigoplus_{\mu\in\mathfrak{h}^*} M_\Theta(\lambda(u))_\mu \tag{3.1.40}$$

and each weight $\mu$ is of the form $\lambda^{(1)} - \omega$, where $\omega$ is a $\mathbb{Z}^+$-linear combination of positive roots in $\Phi_\Theta^+$. Indeed, recalling the PBW Theorem 2.4.5 for $\mathrm{X}(\mathfrak{osp}_{M|N})$, we fix a total order '$\preceq$' on the set $\mathsf{X}$ in such a way that for any $T_{i_1j_1}^{(n_1)}, T_{i_2j_2}^{(n_2)}, T_{i_3j_3}^{(n_3)} \in \mathsf{X}$ satisfying $(i_1,j_1) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^-$, $(i_2,j_2) \in \mathcal{B}_{M|N} \cap \Lambda^\circ$, and $(i_3,j_3) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^+$, then $T_{i_1j_1}^{(n_1)} \preceq T_{i_2j_2}^{(n_2)} \preceq T_{i_3j_3}^{(n_3)}$. Therefore, by the total ordering on $\mathsf{X}$ and Proposition 3.1.4, we conclude that $M_\Theta(\lambda(u))$ is spanned by ordered elements of the form

$$T_{i_1j_1}^{(n_1)} \cdots T_{i_kj_k}^{(n_k)} \mathbf{1}_{\lambda(u)}, \tag{3.1.41}$$

where $k \in \mathbb{N}$, $(i_p, j_p) \in \mathcal{B}_{M|N} \cap \Lambda_\Theta^-$, and $n_p \in \mathbb{Z}^+$ for $1 \le p \le k$. In particular, we conclude that $M_\Theta(\lambda(u))_{\lambda^{(1)}}$ is 1-dimensional; i.e., $M_\Theta(\lambda(u))_{\lambda^{(1)}} = \mathrm{span}_{\mathbb{C}}\{\mathbf{1}_{\lambda(u)}\}$.

Any submodule $P$ of $M_\Theta(\lambda(u))$ also has a weight space decomposition $P = \bigoplus_{\mu\in\mathfrak{h}^*} P_\mu$, where $P_\mu = P \cap M_\Theta(\lambda(u))_\mu$. Since $\dim M_\Theta(\lambda(u))_{\lambda^{(1)}} = 1$, it necessarily follows that $P \subseteq \bigoplus_{\lambda^{(1)}\neq\mu\in\mathfrak{h}^*} M_\Theta(\lambda(u))_\mu$ and so the sum of all proper submodules $K = \sum_{P<M_\Theta(\lambda(u))} P$ is the unique maximal submodule of $M_\Theta(\lambda(u))$.

**Definition 3.1.13.** When the $X_\Theta$-Verma module $M_\Theta(\lambda(u))$ is non-trivial, we define the *irreducible $X_\Theta$-highest weight representation* $L_\Theta(\lambda(u))$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$ with $X_\Theta$-highest weight $\lambda(u)$ as the quotient of the $X_\Theta$-Verma module $M_\Theta(\lambda(u))$ by its unique maximal proper submodule.

As noted prior, the caveat in the definition of Verma modules of $\mathrm{X}(\mathfrak{osp}_{M|N})$ is that they are not yet guaranteed to be non-trivial. In fact, we will see in the next section that there are necessary and sufficient conditions on the highest weight $\lambda(u)$ in order for the Verma module $M_{\mathrm{I}}(\lambda(u))$ to be non-trivial. However, we conclude this section by proving a proposition that describes how one can always modify a collection $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ of formal series of the form (3.1.10) to attain a non-trivial Verma module.

**Proposition 3.1.14.** *Let $\Theta$ denote either* I *or* II. *From any tuple $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ of formal series of the form* (3.1.10), *one can construct a tuple $\widetilde{\lambda}(u)$ of formal series of the same type*

$$\widetilde{\lambda}(u) = (\widetilde{\lambda}_k(u))_{k=1}^{M+N} \quad \textit{such that} \quad \widetilde{\lambda}_b(u) = \lambda_b(u) \quad \textit{for} \quad (b,b) \in \mathcal{B}_{M|N} \ \ \textit{as in} \ \ (2.3.15)$$

*which provides a non-trivial Verma module $M_\Theta(\widetilde{\lambda}(u))$. Furthermore, if $\Theta = \mathrm{I}$ then $\widetilde{\lambda}_M(u) = \lambda_M(u)$, and if $\Theta = \mathrm{II}$ then $\widetilde{\lambda}_{M+N}(u) = \lambda_{M+N}(u)$.*

*Proof.* Let $\mathcal{J}_\Theta(\lambda(u))$ denote the left graded ideal of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of the series $T_{ij}(u)$ for $(i,j) \in \Lambda_\Theta^+ \cap \mathcal{B}_{M|N}$, $T_{kk}(u) - \lambda_k(u)\mathbf{1}$ for $(k,k) \in \mathcal{B}_{M|N}$, and $\mathcal{Z}(u) - \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1}$ if $\Theta = \mathrm{I}$ or $\mathcal{Z}(u) - \lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)\mathbf{1}$ if $\Theta = \mathrm{II}$. We can therefore consider the quotient

$$\widetilde{M}_\Theta(\lambda(u)) := \mathrm{X}(\mathfrak{osp}_{M|N})/\mathcal{J}_\Theta(\lambda(u))$$

By Corollary 2.4.5, choose a total ordering '$\preceq$' on the set $\mathsf{X}$ such that $T_{ij}^{(m)} \preceq T_{aa}^{(b)} \preceq T_{kl}^{(n)}$ for indices $(i,j) \in \Lambda_\Theta^- \cap \mathcal{B}_{M|N}$, $(a,a) \in \mathcal{B}_{M|N}$, $(k,l) \in \Lambda_\Theta^+ \cap \mathcal{B}_{M|N}$ and $m,n \in \mathbb{Z}^+$, $b \in \mathbb{N}$. By such, $\widetilde{M}_\Theta(\lambda(u))$ is spanned by ordered monomials of the form $T_{i_1j_1}^{(m_1)} \cdots T_{i_sj_s}^{(m_s)} \tilde{\mathbf{1}}_{\lambda(u)}$, where $(i_p,j_p) \in \Lambda_\Theta^- \cap \mathcal{B}_{M|N}$ and $m_p \in \mathbb{Z}^+$ for $1 \le p \le s$, and $\tilde{\mathbf{1}}_{\lambda(u)}$ is the image of $\mathbf{1}$ in the quotient $\widetilde{M}_\Theta(\lambda(u))$.

Indeed, let $\mathfrak{N}_\Theta^-$ be the sub-superalgebra of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{ij}(u)$ for $(i,j) \in \Lambda_\Theta^-$; and accordingly let $\mathfrak{B}_\Theta$ be the sub-superalgebra of $\mathrm{X}(\mathfrak{osp}_{M|N})$ generated by the coefficients of $T_{ij}(u)$ for $(i,j) \in \Lambda_\Theta^+ \cup \Lambda^\circ$ and $\mathcal{Z}(u)$. Given the decomposition $\mathfrak{osp}_{M|N} = \mathfrak{n}_\Theta^- \oplus \mathfrak{b}_\Theta$, the images of the sub-superalgebras $\mathfrak{U}(\mathfrak{n}_\Theta^-[z])$ and $\mathfrak{U}((\mathfrak{z}_\mathfrak{c} \oplus \mathfrak{b}_\Theta)[z])$ under the isomorphism (2.4.6) coincides with the respective associated graded superalgebras $\mathrm{gr}\,\mathfrak{N}_\Theta^-$ and $\mathrm{gr}\,\mathfrak{B}_\Theta$ with respect to the filtrations induced by the filtration $\mathbf{E}$ on $\mathrm{X}(\mathfrak{osp}_{M|N})$ via (2.2.21). Accordingly, one can construct appropriate PBW bases for $\mathfrak{N}_\Theta^-$ and $\mathfrak{B}_\Theta$ that show $\mathrm{X}(\mathfrak{osp}_{M|N}) \cong \mathfrak{N}_\Theta^- \otimes \mathfrak{B}_\Theta$.

There is a 1-dimensional representation $\mathbb{C}_{\lambda(u)}$ of $\mathfrak{B}_\Theta$ determined by the actions $T_{ij}(u)\cdot 1 = 0$ for $(i,j)\in\Lambda^+_\Theta\cap\mathcal{B}_{M|N}$, $T_{kk}(u)\cdot 1 = \lambda_k(u)$ for $(k,k)\in\mathcal{B}_{M|N}$, and $\mathcal{Z}(u)\cdot 1 = \lambda_1(u+\kappa)\lambda_M(u)$ if $\Theta=\mathrm{I}$ or $\mathcal{Z}(u)\cdot 1 = \lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)$ if $\Theta = \mathrm{II}$. Via the PBW Theorem for $\mathrm{X}(\mathfrak{osp}_{M|N})$, we can therefore construct the $\mathrm{X}(\mathfrak{osp}_{M|N})$-module

$$\widehat{M}_\Theta(\lambda(u)) := \mathrm{X}(\mathfrak{osp}_{M|N})\otimes_{\mathfrak{B}_\Theta}\mathbb{C}_{\lambda(u)}$$

As a module over $\mathfrak{N}^-_\Theta$, we observe that $\widehat{M}_\Theta(\lambda(u))\cong\mathfrak{N}^-_\Theta$; in particular, it is non-trivial. Finally, the PBW Theorem for $\mathrm{X}(\mathfrak{osp}_{M|N})$ infers there is a module isomorphism $\widetilde{M}_\Theta(\lambda(u))\cong\widehat{M}_\Theta(\lambda(u))$, thus showing $\widetilde{M}_\Theta(\lambda(u))$ is non-trivial as well.

We shall now show that $\widetilde{M}_\Theta(\lambda(u))$ can be realized as an $X_\Theta$-Verma module $M_\Theta(\widetilde{\lambda}(u))$ for some highest weight $\widetilde{\lambda}(u)$. Via the embedding (2.4.7), $\mathrm{X}(\mathfrak{osp}_{M|N})$ may be regarded as an $\mathfrak{osp}_{M|N}$-module whose action described by (3.1.3). By the PBW Theorem and the action of the Cartan subalgebra $\mathfrak{h}$ via (3.1.4), there is a root lattice decomposition $\mathrm{X}(\mathfrak{osp}_{M|N}) = \bigoplus_{\alpha\in\mathbb{Z}\Phi}\mathrm{X}(\mathfrak{osp}_{M|N})_\alpha$, where each generator $T^{(n)}_{ij}$, $(i,j)\in\Lambda^+_\Theta$, will lie in the root space $\mathrm{X}(\mathfrak{osp}_{M|N})_{\alpha_{ij}}$ for some positive root $\alpha_{ij}\in\Phi^+_\Theta$. Writing each such generator $T^{(n)}_{ij}$ as a unique linear combination of PBW basis elements with respect to the total order '$\preceq$', say $T^{(n)}_{ij} = \sum_k\sum_{m_1,\dots,m_k}\gamma_{m_1,\dots,m_k}X_{m_1}\dots X_{m_k}$, then each monomial $X_{m_1}\dots X_{m_k}$ must also lie in $\mathrm{X}(\mathfrak{osp}_{M|N})_{\alpha_{ij}}$.

Since $\alpha_{ij}$ is positive, it is necessary that the the last term in each ordered monomial $X_{m_1}\dots X_{m_k}$ in the expression of $T^{(n)}_{ij}$ is equal to $T^{(m)}_{kl}$ for some $(k,l)\in\Lambda^+_\Theta\cap\mathcal{B}_{M|N}$ and $m\in\mathbb{Z}^+$ by definition of the total order '$\preceq$'. Thus, since $\tilde{\mathbf{1}}_{\lambda(u)}$ is annihilated by each monomial in the expression of $T^{(n)}_{ij}$, then $\tilde{\mathbf{1}}_{\lambda(u)}$ is annihilated by $T^{(n)}_{ij}$ as well.

Similarly, each generator $T^{(b)}_{aa}$, $(a,a)\in\Lambda^\circ$, will lie in the root space $\mathrm{X}(\mathfrak{osp}_{M|N})_0$ (where 0 refers to the zero functional in $\mathfrak{h}^*$). Writing each such generator $T^{(b)}_{aa}$ as a unique linear combination of PBW basis elements with respect to the total order '$\preceq$', say $T^{(n)}_{aa} = \sum_k\sum_{m_1,\dots,m_k}\gamma^\circ_{m_1,\dots,m_k}X_{m_1}\dots X_{m_k}$, then each monomial $X_{m_1}\dots X_{m_k}$ must also lie in $\mathrm{X}(\mathfrak{osp}_{M|N})_0$.

By the definition of the total order '$\preceq$' and the fact that the central series $\mathcal{Z}(u)$ acts on $\tilde{\mathbf{1}}_{\lambda(u)}$ by a scalar series, the action of each ordered monomial $X_{m_1}\dots X_{m_k}$ in the expression of $T^{(b)}_{aa}$ on $\tilde{\mathbf{1}}_{\lambda(u)}$ is of the form

$$\gamma_{m_1,\dots,m_k}\big(T^{(n_1)}_{i_1j_1}\cdots T^{(n_r)}_{i_rj_r}\big)\big(T^{(d_1)}_{c_1c_1}\cdots T^{(d_e)}_{c_ec_e}\big)\big(T^{(m_1)}_{k_1l_1}\cdots T^{(m_s)}_{k_sl_s}\big)\tilde{\mathbf{1}}_{\lambda(u)},$$

where $r, s \in \mathbb{N}$, $(i_q, j_q) \in \Lambda_\Theta^- \cap \mathcal{B}_{M|N}$, $(c_f, c_f) \in \Lambda^\circ \cap \mathcal{B}_{M|N}$, $(k_p, l_p) \in \Lambda_\Theta^+ \cap \mathcal{B}_{M|N}$, and $\gamma_{m_1,\dots,m_k} \in \mathbb{C}$. In particular, since $T_{k_1 l_1}^{(m_1)} \cdots T_{k_s l_s}^{(m_s)}$ lies in $\mathrm{X}(\mathfrak{osp}_{M|N})_\beta$ for some $\beta \in \mathbb{Z}^+\Phi_\Theta^+$, it must be that $T_{i_1 j_1}^{(n_1)} \cdots T_{i_r j_r}^{(n_r)}$ lies in $\mathrm{X}(\mathfrak{osp}_{M|N})_{-\beta}$. Hence, since $\tilde{\mathbf{1}}_{\lambda(u)}$ is an eigenvector for each ordered monomial in the expression of $T_{aa}^{(b)}$, then $\tilde{\mathbf{1}}_{\lambda(u)}$ is an eigenvector for $T_{aa}^{(b)}$ as well. Therefore, for each $(k,k) \in \Lambda^\circ$, we can write $T_{kk}(u)\tilde{\mathbf{1}}_{\lambda(u)} = \widetilde{\lambda}_k(u)\tilde{\mathbf{1}}_{\lambda(u)}$ for some formal power series $\widetilde{\lambda}_k(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, where $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $(k,k) \in \mathcal{B}_{M|N}$.

Defining the ideal $\mathcal{I}_\Theta(\widetilde{\lambda}(u))$ as in Definition 3.1.12, the above argument shows $\mathcal{I}_\Theta(\widetilde{\lambda}(u)) \subseteq \mathcal{J}_\Theta(\lambda(u))$. To prove the reverse inclusion, all that is left to show is that $\mathcal{Z}(u) - \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1}$ lies in $\mathcal{I}_{\mathrm{I}}(\widetilde{\lambda}(u))$ and $\mathcal{Z}(u) - \lambda_{M+1}(u+\kappa)\lambda_{M+N}(u)\mathbf{1}$ lies in $\mathcal{I}_{\mathrm{II}}(\widetilde{\lambda}(u))$. First supposing $\Theta = \mathrm{I}$, setting $i = j = M$ in equation (2.2.18) yields

$$
\begin{aligned}
&\mathcal{Z}(u) - \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1} \\
&\quad = T_{11}(u+\kappa)T_{MM}(u) - \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1} + \sum_{k\neq M}(-1)^{[k]}T_{\bar{k}1}(u+\kappa)T_{kM}(u) \\
&\quad = \big(T_{11}(u+\kappa) - \lambda_1(u+\kappa)\mathbf{1}\big)\big(T_{MM}(u) - \lambda_M(u)\mathbf{1}\big) + \lambda_1(u+\kappa)\big(T_{MM}(u) - \lambda_M(u)\mathbf{1}\big) \\
&\qquad\quad + \lambda_M(u)\big(T_{11}(u+\kappa) - \lambda_1(u+\kappa)\mathbf{1}\big) + \sum_{k\neq M}(-1)^{[k]}T_{\bar{k}1}(u+\kappa)T_{kM}(u),
\end{aligned}
$$

which lies in $\mathcal{I}_{\mathrm{I}}(\widetilde{\lambda}(u))$. The case for $\Theta = \mathrm{II}$ is similar. Hence, $\widetilde{M}_\Theta(\lambda(u)) = M_\Theta(\widetilde{\lambda}(u))$.

Lastly, via the definition of $\mathcal{J}_{\mathrm{I}}(\lambda(u))$, we know $\mathcal{Z}(u)\mathbf{1}_{\widetilde{\lambda}(u)} = \lambda_1(u+\kappa)\lambda_M(u)\mathbf{1}_{\widetilde{\lambda}(u)}$. At the same time, we know by Proposition 3.1.4, that $\mathcal{Z}(u)\mathbf{1}_{\widetilde{\lambda}(u)} = \lambda_1(u+\kappa)\widetilde{\lambda}_M(u)\mathbf{1}_{\widetilde{\lambda}(u)}$; hence, $\widetilde{\lambda}_M(u) = \lambda_M(u)$. A similar argument can be made when $\Theta = \mathrm{II}$ to conclude $\widetilde{\lambda}_{M+N}(u) = \lambda_{M+N}(u)$. □

## 3.2 Finite-Dimensional Irreducible Representations

In this section, we prove our main results on the representation theory of $\mathrm{X}(\mathfrak{osp}_{M|N})$. Ultimately, every finite-dimensional irreducible representation of these Yangians is isomorphic to $L_{\mathrm{I}}(\lambda(u))$ for a certain $X_{\mathrm{I}}$-highest weight $\lambda(u)$. In particular, the highest weight representation theory that follows is solely based on the $X_{\mathrm{I}}$-highest weight theory. Accordingly, we shall drop the prefix '$X_{\mathrm{I}}$' and subscript 'I' for now on, referring to highest weights as $X_{\mathrm{I}}$-highest weights, highest weight vectors as $X_{\mathrm{I}}$-highest weight vectors, $\Lambda^+ = \Lambda_{\mathrm{I}}^+$, $M(\lambda(u)) = M_{\mathrm{I}}(\lambda(u))$, etcetera.

### 3.2.1 Consistency conditions for Verma modules

Given any highest weight representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$, we will now establish a wide array of relations that occur among its highest weight components. Such relations have already been classified in the cases $M=1$ and $M=2$ in [Mol23b] and [Mol22b], respectively. Utilizing the restriction functor $\mathcal{F}^+$ defined in the previous section, we can use inductive arguments along with the results in the cited papers to yield consistency conditions between the highest weight components in the general case.

**Proposition 3.2.1.** *Suppose $N \geq 2$ and let $m = \lfloor \frac{M}{2} \rfloor$, $\widehat{m} = \lceil \frac{M}{2} \rceil$, $n = \frac{N}{2}$. Given any highest weight representation $V$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$, the components of its highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ must satisfy the consistency conditions*

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{\lambda_{M-i}(u-\kappa+i)}{\lambda_{M+1-i}(u-\kappa+i)} \quad \textit{for} \quad i = 1,2,\dots,m-1, \tag{3.2.1}$$

$$\frac{\lambda_{M+j}(u)}{\lambda_{M+j+1}(u)} = \frac{\lambda_{M+N-j}(u-\kappa-j+m)}{\lambda_{M+N+1-j}(u-\kappa-j+m)} \quad \textit{for} \quad j = 1,2,\dots,n-1. \tag{3.2.2}$$

*Moreover, when $M$ is odd:*

$$\frac{\lambda_m(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}(u-\kappa+m)}{\lambda_{\widehat{m}+1}(u-\kappa+m)} \quad \textit{if} \ M \geq 3, \tag{3.2.3}$$

$$\textit{and} \quad \frac{\lambda_{\widehat{m}}(u)}{\lambda_{M+n}(u)} = \frac{\lambda_{M+n+1}(u-\kappa+m-n)}{\lambda_{\widehat{m}}(u-\kappa+m-n)}, \tag{3.2.4}$$

*and when $M$ is even:*

$$\frac{\lambda_m(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}(u-\kappa+m)}{\lambda_{m+1}(u-\kappa+m)} \quad \textit{if} \ M \geq 2. \tag{3.2.5}$$

*Proof.* We shall prove the consistency conditions via induction on $M \in 2\mathbb{Z}^+ - 1$ and $M \in 2\mathbb{Z}^+$.

For the base case $M=1$, consistency conditions were found in [Mol23b] for the presentation $\mathrm{X}^{\mathbf{d}}(\mathfrak{osp}_{1|N})$, where $\mathbf{d} = \{1+\frac{N}{2}\} \subset \{1,2,\dots,1+N\}$. For the bijection $\sigma \in \mathfrak{S}_{1+N}$, where $\sigma(k) = 1+k$ for $k = 1,\dots,\frac{N}{2}$, $\sigma(1+\frac{N}{2}) = 1$, and $\sigma(k) = k$ for $k = 2+\frac{N}{2},\dots,1+N$, the mapping $T^{\mathbf{d}}_{ij}(u) \mapsto T_{\sigma(i)\sigma(j)}(u)$ induces an isomorphism $\mathrm{X}^{\mathbf{d}}(\mathfrak{osp}_{1|N}) \xrightarrow{\sim} \mathrm{X}(\mathfrak{osp}_{1|N})$. Under such isomorphism, highest weight representations for $X^{\mathbf{d}}(\mathfrak{osp}_{1|N})$ defined in the article [Mol23b, §3] coincide with highest weight representations for $X(\mathfrak{osp}_{1|N})$, and the consistency conditions stated in [Mol23b, Proposition 3.3] are equivalent to (3.2.2) and (3.2.4) when $M = 1$.

For the base case $M = 2$, consistency conditions were found in [Mol22b] for the presentation $X^{\mathbf{d}}(\mathfrak{osp}_{2|N})$, where $\mathbf{d} = \{1, 2+N\} \subset \{1, 2, \ldots, 2+N\}$. For the bijection $\sigma \in \mathfrak{S}_{2+N}$, where $\sigma(1) = 1$, $\sigma(k) = 1+k$ for $k = 2, \ldots, 1+N$, and $\sigma(2+N) = 2$, the mapping $T^{\mathbf{d}}_{ij}(u) \mapsto T_{\sigma(i)\sigma(j)}(u)$ induces an isomorphism $X^{\mathbf{d}}(\mathfrak{osp}_{2|N}) \xrightarrow{\sim} X(\mathfrak{osp}_{2|N})$. Under such isomorphism, highest weight representations for $X^{\mathbf{d}}(\mathfrak{osp}_{2|N})$ defined in the article [Mol22b, §2] coincide with highest weight representations for $X(\mathfrak{osp}_{2|N})$, and the consistency conditions stated in [Mol22b, Proposition 2.2] are equivalent to (3.2.2) and (3.2.5) when $M = 2$.

The base case for condition (3.2.3) is when $M = 3$. If $\xi$ denotes the highest weight vector of $V$, we use the relation $T_{14}(u)T_{3,3+N}(v)\xi = \big[T_{14}(u), T_{3,3+N}(v)\big]\xi$ to yield

$$T_{14}(u)T_{3,3+N}(v)\xi = -\frac{1}{u-v-\kappa}\Big(T_{14}(u)T_{3,3+N}(v) + \lambda_4(u)\lambda_{3+N}(v) - \lambda_1(u)\lambda_3(v)\Big)\xi,$$

so $(u-v-\kappa+1)T_{13}(u)T_{3,3+N}(v)\xi = \big(\lambda_4(u)\lambda_{3+N}(v) - \lambda_1(u)\lambda_3(v)\big)\xi$. Setting $v = u-\kappa+1$ then yields the desired relation.

Lastly, the base cases for relations (3.2.1) is when $M = 4$ and $M = 5$, but such relations are guaranteed by Proposition 3.1.5.

Therefore, let us assume the consistency conditions hold up to $M-2$. By Proposition 3.1.6, $V^+$ is a non-trivial $X(\mathfrak{osp}_{(M-2)|N})$-module that contains the highest weight vector $\xi_{\lambda(u)}$, Moreover, the $X(\mathfrak{osp}_{(M-2)|N})$-submodule $X(\mathfrak{osp}_{(M-2)|N})\xi_{\lambda(u)} \subset V^+$ will be a highest weight representation of $X(\mathfrak{osp}_{(M-2)|N})$ with highest weight vector $\xi_{\lambda(u)}$ and highest weight

$$\mu(u) = (\mu_k(u))_{k=1}^{M-2+N} = \big(\lambda_2(u), \ldots, \lambda_{M-1}(u), \lambda_{M+1}(u), \ldots, \lambda_{M+N}(u)\big).$$

Noting the formula $\kappa_{M-2,N} = \kappa_{M|N} - 1$, when $M-2 \geq 4$ our induction hypothesis for $i = 1, 2, \ldots, \lfloor \frac{M-2}{2} \rfloor - 1 = \lfloor \frac{M}{2} \rfloor - 2$ gives

$$\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{\mu_{M-2-i}(u-\kappa_{M-2,N}+i)}{\mu_{M-2+1-i}(u-\kappa_{M-2,N}+i)} \Leftrightarrow \frac{\lambda_{i+1}(u)}{\lambda_{i+2}(u)} = \frac{\lambda_{M-(i+1)}(u-\kappa_{M|N}+(i+1))}{\lambda_{M+1-(i+1)}(u-\kappa_{M|N}+(i+1))},$$

proving the relations (3.2.1) for $i = 2, 3, \ldots, m-1$. The case $i = 1$ is guaranteed by

Proposition 3.1.5. Similarly, we know by induction that for $j = 1, 2, \ldots, n-1$,

$$\frac{\mu_{M-2+j}(u)}{\mu_{M-2+j+1}(u)} = \frac{\mu_{M-2+N-j}(u - \kappa_{M-2,N} - j + \lfloor \frac{M-2}{2} \rfloor)}{\mu_{M-2+N+1-j}(u - \kappa_{M-2,N} - j + \lfloor \frac{M-2}{2} \rfloor)}$$
$$\Leftrightarrow \frac{\lambda_{M+j}(u)}{\lambda_{M+j+1}(u)} = \frac{\lambda_{M+N-j}(u - \kappa_{M|N} - j + \lfloor \frac{M}{2} \rfloor)}{\lambda_{M+N+1-j}(u - \kappa_{M|N} - j + \lfloor \frac{M}{2} \rfloor)}.$$

Now assume $M$ is odd. The induction hypothesis indicates

$$\frac{\mu_{\lceil \frac{M-2}{2} \rceil}(u)}{\mu_{M-2+n}(u)} = \frac{\mu_{M-2+n+1}(u - \kappa_{M-2,N} + \lfloor \frac{M-2}{2} \rfloor - n)}{\mu_{\lceil \frac{M-2}{2} \rceil}(u - \kappa_{M-2,N} + \lfloor \frac{M-2}{2} \rfloor - n)}$$
$$\Leftrightarrow \frac{\lambda_{\lceil \frac{M}{2} \rceil}(u)}{\lambda_{M+n}(u)} = \frac{\lambda_{M+n+1}(u - \kappa_{M|N} + \lfloor \frac{M}{2} \rfloor - n)}{\lambda_{\lceil \frac{M}{2} \rceil}(u - \kappa_{M|N} + \lfloor \frac{M}{2} \rfloor - n)},$$

and if $M-2 \geq 3$,

$$\frac{\mu_{\lfloor \frac{M-2}{2} \rfloor}(u)}{\mu_{M-2+1}(u)} = \frac{\mu_{M-2+N}(u - \kappa_{M-2,N} + \lfloor \frac{M-2}{2} \rfloor)}{\mu_{\lceil \frac{M-2}{2} \rceil + 1}(u - \kappa_{M-2,N} + \lfloor \frac{M-2}{2} \rfloor)} \Leftrightarrow \frac{\lambda_{\lfloor \frac{M}{2} \rfloor}(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}(u - \kappa + \lfloor \frac{M}{2} \rfloor)}{\lambda_{\lceil \frac{M}{2} \rceil + 1}(u - \kappa + \lfloor \frac{M}{2} \rfloor)}.$$

Lastly, if $M$ is even, then

$$\frac{\mu_{\frac{M-2}{2}}(u)}{\mu_{M-2+1}(u)} = \frac{\mu_{M-2+N}(u - \kappa_{M-2,N} + \frac{M-2}{2})}{\mu_{\frac{M-2}{2}+1}(u - \kappa_{M-2,N} + \frac{M-2}{2})} \Leftrightarrow \frac{\lambda_{\frac{M}{2}}(u)}{\lambda_{M+1}(u)} = \frac{\lambda_{M+N}(u - \kappa_{M|N} + \frac{M}{2})}{\lambda_{\frac{M}{2}+1}(u - \kappa_{M|N} + \frac{M}{2})}.$$

□

We now arrive at the primary result of this subsection:

**Theorem 3.2.2.** *Suppose $N \geq 2$. The Verma module $M(\lambda(u))$ is non-trivial if and only if the components of the highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfy the consistency conditions in Proposition 3.2.1. Hence, for every finite-dimensional irreducible representation $V$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$, it holds that*

$$V \cong L(\lambda(u))$$

*for some unique tuple $\lambda(u)$ satisfying such conditions.*

*Proof.* "⇒" If $M(\lambda(u))$ is non-trivial, then it is a highest weight representation. Hence, the consistency conditions follow from Proposition 3.2.1.

"$\Leftarrow$" Conversely, let us suppose the highest weight $\lambda(u)$ satisfies the conditions (3.2.1), (3.2.2), and (3.2.3), (3.2.4) if $M$ is odd or (3.2.5) if $M$ is even. By Proposition 3.1.14, we obtain a non-trivial Verma module $M(\widetilde{\lambda}(u))$ To finish the proof, it therefore suffices to show $\widetilde{\lambda}(u) = \lambda(u)$. As $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $(k, k) \in \mathcal{B}_{M|N}$ as in (2.3.15) and $k = M$, we need to show the equality in the remaining cases.

Furthermore, since $M(\widetilde{\lambda}(u))$ is non-trivial and the first conditional statement of the Proposition has been proven, the highest weight components of $\widetilde{\lambda}(u) = (\widetilde{\lambda}_k(u))_{k=1}^{M+N}$ satisfy the relations

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{\widetilde{\lambda}_{M-1}(u-\kappa+i)}{\lambda_M(u-\kappa+i)} \quad \text{and} \quad \frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{\widetilde{\lambda}_{M-i}(u-\kappa+i)}{\widetilde{\lambda}_{M+1-i}(u-\kappa+i)}, \tag{3.2.6}$$

for $i = 2, \ldots, m-1$,

$$\frac{\lambda_{M+j}(u)}{\lambda_{M+j+1}(u)} = \frac{\widetilde{\lambda}_{M+N-j}(u-\kappa-j+m)}{\widetilde{\lambda}_{M+N+1-j}(u-\kappa-j+m)} \tag{3.2.7}$$

for $j = 1, 2, \ldots, n-1$, and if $M$ is odd:

$$\frac{\lambda_m(u)}{\lambda_{M+1}(u)} = \frac{\widetilde{\lambda}_{M+N}(u-\kappa+m)}{\widetilde{\lambda}_{\widehat{m}+1}(u-\kappa+m)} \quad \text{when } M \geq 3, \tag{3.2.8}$$

$$\text{and} \quad \frac{\widetilde{\lambda}_{\widehat{m}}(u)}{\lambda_{M+n}(u)} = \frac{\widetilde{\lambda}_{M+n+1}(u-\kappa+m-n)}{\widetilde{\lambda}_{\widehat{m}}(u-\kappa+m-n)}, \tag{3.2.9}$$

or if $M$ is even:

$$\frac{\lambda_m(u)}{\lambda_{M+1}(u)} = \frac{\widetilde{\lambda}_{M+N}(u-\kappa+m)}{\widetilde{\lambda}_{m+1}(u-\kappa+m)}. \tag{3.2.10}$$

When $M = 1$, equations (3.2.4) and (3.2.9) yield $\widetilde{\lambda}_{M+n+1}(u) = \lambda_{M+n+1}(u)$. Thus, by combining (3.2.2) and (3.2.7), we obtain $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = M+n+2, \ldots, M+N$.

When $M = 2$, (3.2.5) and (3.2.10) infer $\widetilde{\lambda}_{M+N}(u) = \lambda_{M+N}(u)$. Therefore, combining (3.2.2) with (3.2.7) shows $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = M+n+1, \ldots, M+N-1$.

When $M = 3$, (3.2.3) and (3.2.8) similarly infer $\widetilde{\lambda}_{M+N}(u) = \lambda_{M+N}(u)$. Hence, combining (3.2.2) with (3.2.7) gives $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = M+n+1, \ldots, M+N-1$.

Now assume $M \geq 4$. In this case, relations (3.2.1) and (3.2.6) show $\widetilde{\lambda}_k(u) = \lambda_k(u)$

for $k = \widehat{m}+1, \dots, M-1$. Furthermore, by combining (3.2.3) and (3.2.8) if $M$ is odd, or (3.2.5) and (3.2.10) if $M$ is even, one deduces $\widetilde{\lambda}_{M+N}(u) = \lambda_{M+N}(u)$. Thus, combining (3.2.2) and (3.2.7) will show $\widetilde{\lambda}_k(u) = \lambda_k(u)$ for $k = M+n+1, \dots, M+N-1$. Finally, when $M$ is odd, relations (3.2.4) and (3.2.9) will together yield the last equality $\widetilde{\lambda}_{\widehat{m}}(u) = \lambda_{\widehat{m}}(u)$. □

### 3.2.2 Finite-dimensional irreducible representations

Let us assume $m = \lfloor \frac{M}{2} \rfloor$ and $n = \frac{N}{2}$. For any tuple $\lambda = (\lambda_1, \dots, \lambda_{m+n}) \in \mathbb{C}^{m+n}$, we shall let $V(\lambda)$ denote the irreducible representation of the orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|N}$ with highest weight $\lambda$, where we suppose $M \geq 3$. Most necessary conditions for the finite-dimensionality of $V(\lambda)$ can be derived from the embeddings $\mathfrak{so}_M \hookrightarrow \mathfrak{osp}_{M|N}$ and $\mathfrak{sp}_N \hookrightarrow \mathfrak{osp}_{M|N}$. In particular, if $V(\lambda)$ is finite-dimensional then it must be that

$$\lambda_i - \lambda_{i+1} \in \mathbb{N} \quad \text{for} \quad i = 1, \dots, m-1;\ m+1, \dots, m+n-1,$$

along with

$$\begin{aligned} & \lambda_{m+n} \in \mathbb{N}, & \\ \text{and} \quad & \lambda_{m-1} + \lambda_m \in \mathbb{N} & \text{if } M \text{ is even,} \\ \text{or} \quad & 2\lambda_m \in \mathbb{N} & \text{if } M \text{ is odd.} \end{aligned}$$

A weight $\lambda \in \mathbb{C}^{m+n}$ satisfying these conditions will be called $\Phi^+_{\text{even}}$-*dominant integral.* Since $\Phi^+$ is not the distinguished positive root system $\Phi^+_{\text{dist}}$ as found in [Kac06], one would have to translate between them by means of odd and/or real reflections in order to state all necessary and sufficient conditions for the finite-dimensionality of $V(\lambda)$ in terms of $\Phi^+$-highest weights.

Before stating the main theorem, we recall the super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and its representation theory as appeared in [Zha96]. For the following results on this particular Yangian, we suppose $m$ and $n$ are any natural numbers such that $m+n \geq 1$ and consider the gradation index (2.1.5) when $d = m$, $D = m+n$, and $\mathbf{d} = \{1, 2, \dots, m\}$; that is, $[i] = \bar{0}$ for $1 \leq i \leq m$ and $[i] = \bar{1}$ for $m+1 \leq i \leq m+n$. Accordingly, the space $\mathbb{C}^{m+n}$ is $\mathbb{Z}_2$-graded via the assignment $[e_i] = [i]$ on the standard basis vectors and we shall denote the resulting super vector space as $\mathbb{C}^{m|n}$. The simplest non-trivial solution to the SQYBE (2.2.4) in the space $\left(\operatorname{End}\mathbb{C}^{m|n}\right)^{\otimes 3}[\![u^{\pm 1}v^{\pm 1}]\!]$ is the $R$-matrix

$$\dot{R}(u) = \mathrm{id}^{\otimes 2} - \frac{\dot{P}}{u} \in \left(\operatorname{End}\mathbb{C}^{m|n}\right)^{\otimes 2}[u^{-1}], \tag{3.2.11}$$

where $\dot{P} = \sum_{i,j=1}^{m+n}(-1)^{[j]}E_{ij}\otimes E_{ji}$ is the super permutation operator.

**Definition 3.2.3.** The *Yangian* $\mathrm{Y}(\mathfrak{gl}_{m|n})$ of $\mathfrak{gl}_{m|n}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\left\{t_{ij}^{(r)} \mid 1 \le i,j \le m+n,\ r\in\mathbb{Z}^+\right\}$, with $\mathbb{Z}_2$-grade $\left[t_{ij}^{(r)}\right] := [i]+[j]$ for all $r\in\mathbb{Z}^+$, subject to the defining $RTT$-relation

$$\begin{aligned} &\dot{R}(u-v)t_1(u)t_2(v) = t_2(v)t_1(u)\dot{R}(u-v) \\ \text{in}\quad &\left(\operatorname{End}\mathbb{C}^{m|n}\right)^{\otimes 2}\otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[\![u^{\pm 1},v^{\pm 1}]\!], \end{aligned}$$

where $t(u) := \sum_{i,j=1}^{m+n}(-1)^{[i][j]+[j]}E_{ij}\otimes t_{ij}(u) \in \operatorname{End}(\mathbb{C}^{m|n})\otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[\![u^{-1}]\!]$ is the matrix consisting of the series $t_{ij}(u) := \delta_{ij}\mathbf{1}+\sum_{r=1}^{\infty}t_{ij}^{(r)}u^{-r} \in \mathrm{Y}(\mathfrak{gl}_{m|n})[\![u^{-1}]\!]$ for $1\le i,j\le m+n$, and $\dot{R}(u-v)$ is the $R$-matrix (3.2.11) identified with $\dot{R}(u-v)\otimes\mathbf{1}$

We may now state the definition of a highest weight representation for this super Yangian.

**Definition 3.2.4** (See §3.1 in [Zha96]). A representation $V$ of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a *highest weight representation* if there exists a nonzero vector $\xi\in V$ such that $\mathrm{Y}(\mathfrak{gl}_{m|n})\xi = V$, and

$$\begin{aligned} t_{ij}(u)\xi &= 0 &&\text{for all}\quad 1\le i<j\le m+n \\ \text{and}\quad t_{kk}(u)\xi &= \lambda_k(u)\xi &&\text{for all}\quad 1\le k\le m+n, \end{aligned}$$

where $\lambda_k(u) = 1+\sum_{n=1}^{\infty}\lambda_k^{(n)}u^{-n}\in 1+\mathbb{C}[\![u^{-1}]\!]u^{-1}$. We say that $\xi$ is the *highest weight vector* of $V$ and call the $m+n$-tuple $(\lambda_k(u))_{k=1}^{m+n}$ of formal series the *highest weight* of $V$.

The classification of the finite-dimensional irreducible representations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ by R .B. Zhang is provided by the following two theorems:

**Theorem 3.2.5** (Theorem 2 in [Zha96]). *Every finite-dimensional irreducible representation $V$ of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a highest weight representation. The highest weight vector of $V$ is unique up to scalar multiple.*

**Theorem 3.2.6** (Theorem 4 in [Zha96]). *An irreducible highest weight representation $V$ of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ with highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{m+n}$ is finite-dimensional if and only if there exists monic polynomials $\widetilde{Q}(u)$, $Q(u)$, and $P_k(u)$, $k \in \{1, 2, \ldots, m+n-1\} \setminus \{m\}$, such that*

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u + (-1)^{[k]})}{P_k(u)} \quad \textit{for} \quad k \in \{1, 2, \ldots, m+n-1\} \setminus \{m\},$$

*and*

$$\frac{\lambda_m(u)}{\lambda_{m+1}(u)} = \frac{\widetilde{Q}(u)}{Q(u)},$$

*where $\widetilde{Q}(u)$ and $Q(u)$ are coprime polynomials of the same polynomial degree.*

For the remainder of this subsection, we assume $m = \lfloor \frac{M}{2} \rfloor$, $\widehat{m} = \lceil \frac{M}{2} \rceil$, and $n = \frac{N}{2}$. From Molev's recent work (see [Mol21], [Mol23b], and [Mol22b]) on the representation theory of $\mathrm{X}^{\mathbf{d}_1}(\mathfrak{osp}_{1|N})$ and $\mathrm{X}^{\mathbf{d}_2}(\mathfrak{osp}_{2|N})$ where $\mathbf{d}_1 = \{1+n\}$ and $\mathbf{d}_2 = \{1, 2+N\}$, we can use the isomorphisms $\mathrm{X}^{\mathbf{d}_1}(\mathfrak{osp}_{1|N}) \xrightarrow{\sim} \mathrm{X}(\mathfrak{osp}_{1|N})$ and $\mathrm{X}^{\mathbf{d}_2}(\mathfrak{osp}_{2|N}) \xrightarrow{\sim} \mathrm{X}(\mathfrak{osp}_{2|N})$ to get the following theorem:

**Theorem 3.2.7** (A. Molev). *Suppose $M = 1$ or $M = 2$, set $N \geq 2$, and let $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfy the consistency conditions stated in Proposition 3.2.1. The $\mathrm{X}(\mathfrak{osp}_{M|N})$-module $L(\lambda(u))$ is finite-dimensional if and only if there exists a tuple of monic polynomials*

$$(\delta_{M2}\widetilde{Q}(u), \delta_{M2}Q(u); (P_k(u))_{k\in I}) \in \mathbb{C}[u]^{n+2\delta_{M2}},$$

*with $I = \{M+1, \ldots, M+n\}$, such that*

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u-1)}{P_k(u)} \quad \textit{for all} \quad k \in I \setminus \{M+n\},$$

*and*

$$\frac{\lambda_1(u)}{\lambda_{1+n}(u)} = \frac{P_{1+n}(u+1)}{P_{1+n}(u)} \quad \textit{when} \quad M = 1,$$

*or*

$$\frac{\lambda_{2+n}(u)}{\lambda_{2+n+1}(u)} = \frac{P_{2+n}(u-2)}{P_{2+n}(u)} \qquad \textit{and} \qquad \frac{\lambda_1(u)}{\lambda_3(u)} = \frac{\widetilde{Q}(u)}{Q(u)} \quad \textit{when} \quad M = 2,$$

*where $\widetilde{Q}(u)$ and $Q(u)$ are coprime polynomials of the same polynomial degree.*

Via Theorem 3.2.2, every finite-dimensional irreducible representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ is isomorphic to $L(\lambda(u))$ for some highest weight $\lambda(u)$ satisfying the consistency conditions in Proposition 3.2.1. We now have the main theorem of this subsection:

**Theorem 3.2.8.** *Suppose $M, N \geq 2$ and let $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ satisfy the consistency conditions stated in Proposition 3.2.1. If the $\mathrm{X}(\mathfrak{osp}_{M|N})$-module $L(\lambda(u))$ is finite-dimensional, then there exists a tuple of monic polynomials*

$$(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I}) \in \mathbb{C}[u]^{m+n+1},$$

*with $I = \{1, \ldots, m-1; M+1, \ldots, M+n\}$, such that*

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u + (-1)^{[k]})}{P_k(u)} \qquad \text{for} \quad k \in I \setminus \{M+n\}, \tag{3.2.12}$$

$$\frac{\lambda_{\widehat{m}}(u)}{\lambda_{M+n}(u)} = \frac{P_{M+n}(u+1)}{P_{M+n}(u)} \quad \text{if} \quad M \text{ is odd}, \tag{3.2.13}$$

$$\frac{\lambda_{M+n}(u)}{\lambda_{M+n+1}(u)} = \frac{P_{M+n}(u-2)}{P_{M+n}(u)} \quad \text{if} \quad M \text{ is even}, \tag{3.2.14}$$

*and*

$$\frac{\lambda_m(u)}{\lambda_{M+1}(u)} = \frac{\widetilde{Q}(u)}{Q(u)}, \tag{3.2.15}$$

*where $\widetilde{Q}(u)$ and $Q(u)$ are coprime polynomials of the same polynomial degree.*

The polynomials $(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$ are called the *Drinfel'd polynomials* corresponding to $L(\lambda(u))$ and they are uniquely determined by the highest weight $\lambda(u)$.

*Proof of Theorem 3.2.8.* Allowing $t_{ij}(u)$ denote a generating series for the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$, there is a superalgebra morphism

$$\nu\colon \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{X}(\mathfrak{osp}_{M|N}), \quad t_{ij}(u) \mapsto T_{\nu(i)\nu(j)}(u),$$

where $\nu(i) = i$ for $1 \leq i \leq m$ and $\nu(i) = \widehat{m}+i$ for $m+1 \leq i \leq m+n$. Under the above morphism $\nu$, representations of $\mathrm{X}(\mathfrak{osp}_{M|N})$ restrict to those of $\mathrm{Y}(\mathfrak{gl}_{m|n})$. In particular, if $V$ is a highest weight representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ with highest weight vector $\xi$ and highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$, then the submodule $\mathrm{Y}(\mathfrak{gl}_{m|n})\xi \subset V$ will be a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ with the highest weight vector $\xi$ and highest weight $\big(\lambda_1(u), \dots, \lambda_m(u), \lambda_{M+1}(u), \dots, \lambda_{M+n}(u)\big)$.

Assuming that the irreducible highest weight representation $L(\lambda(u))$ is finite-dimensional, the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $\mathrm{Y}(\mathfrak{gl}_{m|n})\mathbf{1}_{\lambda(u)}$ will be a finite-dimensional as well. The quotient $\mathrm{Y}(\mathfrak{gl}_{m|n})\mathbf{1}_{\lambda(u)}/\mathcal{M}$, where $\mathcal{M}$ is its maximal submodule, will be an irreducible finite-dimensional representation of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ with highest weight vector $\mathbf{1}_{\lambda(u)} \bmod \mathcal{M}$ with highest weight $\big(\lambda_1(u), \dots, \lambda_m(u), \lambda_{M+1}(u), \dots, \lambda_{M+n}(u)\big)$. Therefore, the Drinfel'd polynomial relations for irreducible finite-dimensional representations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ yield the according relations (3.2.12) and (3.2.15).

The remaining Drinfel'd polynomial relations will be proved via induction on $M \in 2\mathbb{Z}^+ - 1$ and $M \in 2\mathbb{Z}^+$, respectively. Let us first suppose $M$ is odd. The base case $M = 1$ is guaranteed by Molev's results in [Mol23b]. Applying the restriction functor of Proposition 3.1.9 to $L(\lambda(u))$, we yield an $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})$-submodule $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})\mathbf{1}_{\lambda(u)} \subset \mathcal{F}^+\big(L(\lambda(u))\big) = L(\lambda(u))^+$ that is a highest weight representation with highest weight vector $\mathbf{1}_{\lambda(u)}$ and highest weight

$$\big(\mu_k(u)\big)_{k=1}^{M-2+N} = \big(\lambda_2(u), \dots, \lambda_{M-1}(u), \lambda_{M+1}(u), \dots, \lambda_{M+N}\big).$$

Since $L(\lambda(u))$ is finite-dimensional, then so is $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})\mathbf{1}_{\lambda(u)}$ and the quotient $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})\mathbf{1}_{\lambda(u)}/\mathcal{M}$ by its maximal submodule $\mathcal{M}$. Such irreducible quotient will also be a highest weight representation with highest weight vector $\mathbf{1}_{\lambda(u)} \bmod \mathcal{M}$ and the same highest weight as above.

Hence, by induction hypothesis there exists a monic polynomial $P^{\mu}_{M-2+n}(u)$ such that

$$\frac{\lambda_{\lceil \frac{M}{2} \rceil}(u)}{\lambda_{M+n}(u)} = \frac{\mu_{\lceil \frac{M-2}{2} \rceil}(u)}{\mu_{M-2+n}(u)} = \frac{P^{\mu}_{M-2+n}(u+1)}{P^{\mu}_{M-2+n}(u)},$$

so set $P_{M+n}(u) = P^{\mu}_{M-2+n}(u)$.

When $M$ is even, the base $M = 2$ is provided by Molev's work in [Mol22b]. The same argument to the above shows that one can construct a finite-dimensional irreducible highest weight representation $\mathrm{X}(\mathfrak{osp}_{(M-2)|N})\mathbf{1}_{\lambda(u)}/\mathcal{M}$ with highest weight vector $\mathbf{1}_{\lambda(u)} \bmod \mathcal{M}$ and highest weight $\big(\mu_k(u)\big)_{k=1}^{M-2+N}$. Hence, by induction hypothesis there exists a monic polynomial $P^{\mu}_{M-2+n}(u)$ such that

$$\frac{\lambda_{M+n}(u)}{\lambda_{M+n+1}(u)} = \frac{\mu_{M-2+n}(u)}{\mu_{M-2+n+1}(u)} = \frac{P^{\mu}_{M-2+n}(u-2)}{P^{\mu}_{M-2+n}(u)},$$

so set $P_{M+n}(u) = P^{\mu}_{M-2+n}(u)$. □

Recalling that $\mathrm{Rep}(\mathcal{A})$ denotes the category of representations of a superalgebra $\mathcal{A}$, we let $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathcal{A})$ denote the subcategory of finite-dimensional irreducible representations. Further, we define $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathcal{A})_{/\sim}$ to be the set of isomorphism classes of $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathcal{A})$.

Letting $\mathbb{C}[u]^2_{\mathrm{cp,ed}}$ denote the subset of $\mathbb{C}[u]^2$ consisting of all pairs $(B_1(u), B_2(u))$ such that $B_1(u)$ and $B_2(u)$ are coprime of the same polynomial degree, the proof of Theorem 3.2.8 permits the well-defined map

$$\begin{aligned} \mho\colon \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{X}(\mathfrak{osp}_{M|N}))_{/\sim} &\to \left\{(B_k(u))_{k=1}^{m+n+1} \in \mathbb{C}[u]^2_{\mathrm{cp,ed}} \times \mathbb{C}[u]^{m+n-1} \,|\, B_k(u) \text{ is monic}\right\} \\ L(\lambda(u)) &\mapsto (\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I}) \end{aligned}$$

assuming $M, N \geq 2$. However, such map is not injective: $\mho(L(\lambda(u))) = \mho(L(\mu(u)))$ if and only if there exists a series $f(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that $\mu(u) = f(u)\lambda(u)$.

Indeed, if the tuples $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ and $\mu(u) = (\mu_k(u))_{k=1}^{M+N}$ both satisfy the consistency conditions in Proposition 3.2.1 while also corresponding to the same Drinfel'd polynomials $(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$, then there exists a series $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that $\mu_k(u) = f(u)\lambda_k(u)$ for all $1 \leq k \leq M+N$: such a series is given by $f(u) = \mu_{M+n}(u)\lambda_{M+n}(u)^{-1}$.

On the other hand, given that $L(\lambda(u))$ is finite-dimensional, its highest weight $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ must satisfy the consistency conditions as in Proposition 3.2.1 while also corresponding to the Drinfel'd polynomials $(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$. For any series $f(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$, we observe the tuple $f(u)\lambda(u) = (f(u)\lambda_k(u))_{k=1}^{M+N}$ will still satisfy the same consistency conditions and Drinfel'd polynomials relations. Hence, the pullback by the automorphism $\mu_f$ (2.2.9) will yield a module $\mu_f^*(L(\lambda(u)))$ isomorphic to $L(f(u)\lambda(u))$. As $L(\lambda(u))$ and $\mu_f^*(L(\lambda(u)))$ are equal as super vector spaces, their dimensions coincide, which infers the same is true for $L(f(u)\lambda(u))$, so $\dim L(f(u)\lambda(u)) < \infty$ as well.

We now focus our attention on certain elementary tuples of Drinfel'd polynomials and we call those modules corresponding to such tuples as fundamental representations:

**Definition 3.2.9.** Let $\lambda(u)$ satisfy the consistency conditions as stated in Proposition 3.2.1 so that the Verma module $M(\lambda(u))$ is non-trivial. The *fundamental representations* of $\mathrm{X}(\mathfrak{osp}_{M|N})$ are those irreducible representations $L(\lambda(u))$ that correspond to Drinfel'd polynomials of the form

$$\big(u+\alpha, u+\beta; (1)_{k\in I}\big) \quad \text{or} \quad \big(1, 1; ((u+\gamma)^{\delta_{ik}})_{k\in I}\big) \tag{3.2.16}$$

for $i \in I$ and $\alpha, \beta, \gamma \in \mathbb{C}$ where $\alpha \neq \beta$. The fundamental representations corresponding to the first tuple are called *type* I and denoted $L(\lambda(u); \alpha, \beta)$, whereas those corresponding to the second tuple are called *type* II and denoted $L(\lambda(u); i : \gamma)$.

Assuming we are able to prove that all such fundamental representations are finite-dimensional, then one can construct finite-dimensional irreducible representation of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|N})$ corresponding to *any* tuple of Drinfel'd polynomials by virtue of the following lemma.

**Lemma 3.2.10.** *Let $V$ and $W$ be two highest weight representations of $\mathrm{X}(\mathfrak{osp}_{M|N})$ with respective highest weights $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$, $\mu(u) = (\mu_k(u))_{k=1}^{M+N}$ and highest weight vectors $\xi_{\lambda(u)}$, $\xi_{\mu(u)}$. The submodule $\mathrm{X}(\mathfrak{osp}_{M|N})(\xi_{\lambda(u)} \otimes \xi_{\mu(u)}) \in V \otimes W$ will be a highest weight representation with highest weight $\lambda(u)\mu(u) = (\lambda_k(u)\mu_k(u))_{k=1}^{M+N}$.*

*Further, if $V = L(\lambda(u))$ and $W = L(\mu(u))$ are finite-dimensional with corresponding tuples of Drinfel'd polynomials $(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$ and $(\widetilde{Q}'(u), Q'(u); (P'_k(u))_{k\in I})$, then the tuple of Drinfel'd polynomials corresponding to the irreducible quotient of $\mathrm{X}(\mathfrak{osp}_{M|N})(\mathbf{1}_{\lambda(u)} \otimes \mathbf{1}_{\mu(u)}) \in L(\lambda(u)) \otimes L(\mu(u))$ will be*

$$\Big(\frac{\widetilde{Q}(u)\widetilde{Q}'(u)}{d(u)}, \frac{Q(u)Q'(u)}{d(u)}; (P_k(u)P'_k(u))_{k\in I}\Big) \tag{3.2.17}$$

*where $d(u) = \gcd(\widetilde{Q}(u)\widetilde{Q}'(u), Q(u)Q'(u))$ is monic.*

*Proof.* Via the comultiplication map $\Delta$ on $\mathrm{X}(\mathfrak{osp}_{M|N})$, the generators $T_{ij}(u)$ with indices $(i,j) \in \Lambda^+$ will annihilate $\xi_{\lambda(u)} \otimes \xi_{\mu(u)}$ since for such indices, $(i,k) \notin \Lambda^+$ implies $(k,j) \in \Lambda^+_{\mathrm{I}}$. Once can also verify $T_{kk}(u) \cdot (\xi_{\lambda(u)} \otimes \xi_{\mu(u)}) = \lambda_k(u)\mu_k(u)(\xi_{\lambda(u)} \otimes \xi_{\mu(u)})$ for all integers $1 \leq k \leq M+N$. □

### 3.2.3 Type I fundamental representations

In this subsection, we show that type I fundamental representations $L(\lambda(u);\alpha,\beta)$ as in Definition 3.2.9 are finite-dimensional if and only if the parameters $\alpha$ and $\beta$ satisfy certain conditions.

We note that if a non-trivial irreducible representation $L(\lambda(u))$ has associated Drinfel'd polynomials $(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$, the pullback by the shift automorphism $\tau_a$ (2.2.10) will yield a module $\tau_a^*(L(\lambda(u)))$ isomorphic to $L(\lambda(u-a))$ with Drinfel'd polynomials $(\widetilde{Q}(u-a), Q(u-a); (P_k(u-a))_{k\in I})$. In particular, the dimensions of $L(\lambda(u))$ and $L(\lambda(u-a))$ coincide, so it suffices to prove $\dim L(\lambda(u);\alpha,0) < \infty$ for some highest weight $\lambda(u)$ satisfying the consistency conditions in Proposition 3.2.1.

The primary result of this subsection is the following, which is a generalization of [Mol22b, Proposition 2.4]:

**Proposition 3.2.11.** *Suppose $M,N \geq 2$. For $\alpha \in \mathbb{C}^*$, consider the type I fundamental representation $L(\lambda(u);\alpha,0)$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$, where $\lambda(u)$ is the highest weight $(\lambda_k(u))_{k=1}^{M+N}$ given by*

$$\lambda_k(u) = \begin{cases} \dfrac{u+\alpha}{u} & \text{if } \; 1 \leq k \leq m, \\ \dfrac{u+\kappa-m}{u+\alpha+\kappa-m} & \text{if } \; \widehat{m}+1 \leq k \leq M, \\ 1 & \text{otherwise.} \end{cases}$$

*If $M=2$, then $\dim L(\lambda(u);\alpha,0) \leq 2^N$. Otherwise when $M \geq 3$, then $L(\lambda(u);\alpha,0)$ is finite-dimensional if and only if the irreducible $\mathfrak{osp}_{M|N}$-module $V(\lambda^{(1)})$ is finite-dimensional, where*

$$\lambda^{(1)} := (\lambda_1^{(1)}, \dots, \lambda_m^{(1)}, \lambda_{M+1}^{(1)}, \dots, \lambda_{M+n}^{(1)}) = (\underbrace{\alpha,\dots,\alpha}_{m}, \underbrace{0,\dots,0}_{n}).$$

*The collection of such numbers $\alpha$ comprise some non-trivial subset of $\frac{1}{2}\mathbb{Z}^+$. When $M$ is even in these cases, then $\dim L(\lambda(u);\alpha,0) \leq 2^{mN}$.*

In preparation to prove the above proposition, we need to introduce and prove two preliminary computational lemmas.

**Lemma 3.2.12.** *Suppose $M, N \geq 2$ and let $\xi$ denote the highest weight vector of $L(\lambda(u);\alpha,0)$ as in Proposition 3.2.11. Then:*

(i) $T_{kl}(v)\xi = 0$ *for indices* $M+1 \leq k \neq l \leq M+N$*; moreover, when $M$ is odd:* $T_{k\widehat{m}}(v)\xi = T_{\widehat{m}k}(v)\xi = 0$ *for indices* $M+1 \leq k \leq M+N$,

(ii) $T_{kl}(v)\xi = 0$ *for indices* $1 \leq k \neq l \leq m$ *and* $\widehat{m}+1 \leq k \neq l \leq M$,

(iii) $T^{(r)}_{kl}\xi = 0$ *for indices* $M+1 \leq k \leq M+N$, $1 \leq l \leq m$ *with* $r \geq 2$*; moreover, when $M$ is odd:* $T^{(r)}_{\widehat{m}l}\xi = 0$ *for indices* $1 \leq l \leq m$ *with* $r \geq 2$.

Before proving the above lemma, we note that for a highest weight representation $V$ of $\mathrm{X}(\mathfrak{osp}_{M|N})$ with highest weight vector $\xi$, we call a vector $v \in V$ *singular* if $T_{ij}(u)v = 0$ for all $(i,j) \in \Lambda^+$. Since $L(\lambda(u);\alpha,0)$ is irreducible, its only singular vectors lie in $\mathbb{C}^*\xi$; thus, to prove $T_{kl}(v)\xi = 0$ it suffices to show $T_{ij}(u)T_{kl}(v)\xi = 0$ for all $(i,j) \in \Lambda^+$.

*Proof of Lemma 3.2.12.* (i) Allowing $\mathcal{F}^+_M$ to denote the restriction functor $\mathcal{F}^+$ as in Proposition 3.1.9, we may apply the composition $\mathcal{F}^+_{M-2m+2} \circ \cdots \circ \mathcal{F}^+_{M-2} \circ \mathcal{F}^+_M$ to $L(\lambda(u);\alpha,0)$ to obtain the $\mathrm{X}(\mathfrak{osp}_{(M-2m)|N})$-module $L(\lambda(u);\alpha,0)^{\Sigma_m+}$ as described in Remark 3.1.11. For the given indices $M+1 \leq k, l \leq M+N$, we claim that the vectors $T_{kl}(v)\xi$, along with $T_{k\widehat{m}}(v)\xi$ and $T_{\widehat{m}l}(v)\xi$ if $M$ is odd, lie in $L(\lambda(u);\alpha,0)^{\Sigma_m+}$. Addressing vectors of the form $T_{kl}(v)\xi$, we may suppose $k > l$ without loss of generality. For $1 \leq i \leq m$ and $i < j \leq M+N$, using the defining relations (2.2.8) to compute $\left[T_{ij}(u), T_{kl}(v)\right]$ will yield

$$T_{ij}(u)T_{kl}(v)\xi = \frac{1}{u-v-\kappa}\delta_{\bar{\jmath}l}\sum_{p=1}^{i-1}\theta_j T_{k\bar{p}}(v)T_{ip}(u)\xi,$$

while for indices $1 \leq p \leq i-1$, the evaluation of $\left[T_{ip}(u), T_{k\bar{p}}(v)\right]$ infers the equality $\delta_{\bar{\jmath}l}\theta_j T_{k\bar{p}}(v)T_{ip}(u)\xi = -T_{ij}(u)T_{kl}(v)\xi$, so $T_{ij}(u)T_{kl}(v)\xi = 0$. Alternatively, when we have $m+1 \leq i < j \leq M$ or $M+1 \leq i \leq M+N$, $\widehat{m}+1 \leq j \leq M$, computing $\left[T_{kl}(v), T_{ij}(u)\right]$ will give

$$T_{ij}(u)T_{kl}(v)\xi = \frac{1}{v-u-\kappa}\delta_{\bar{\imath}k}\sum_{p=1}^{\bar{\jmath}-1}\theta_k T_{pl}(v)T_{\bar{p}j}(u)\xi,$$

while for indices $1 \leq p \leq \bar{\jmath}-1$, the evaluation of $\left[T_{pl}(v), T_{\bar{p}j}(u)\right]$ infers the equality $\delta_{\bar{\imath}k}\theta_k T_{pl}(v)T_{\bar{p}j}(u)\xi = -T_{ij}(u)T_{kl}(v)\xi$, so $T_{ij}(u)T_{kl}(v)\xi = 0$, which proves $T_{kl}(v)\xi$ lies in $L(\lambda(u);\alpha,0)^{\Sigma_m+}$.

Since $\xi \in L(\lambda(u);\alpha,0)^{\Sigma_m+}$, the cyclic submodule $\mathrm{X}(\mathfrak{osp}_{(M-2m)|N})\xi$ is a highest weight representation of $\mathrm{X}(\mathfrak{osp}_{(M-2m)|N})$ with each highest weight component $\lambda_k(u)$ equal to 1. The irreducible quotient of $\mathrm{X}(\mathfrak{osp}_{(M-2m)|N})\xi$ is therefore 1-dimensional, which implies $T_{ij}(u)T_{kl}(v)\xi = 0$ for all $(i,j) \in \Lambda^+$. The argument for $T_{k\widehat{m}}(v)\xi = T_{\widehat{m}l}(v)\xi = 0$ when $M$ is odd is similar.

(ii) When $M$ is even, the relations (3.1.2) infer that it suffices to show the property $T_{ij}(u)T_{kl}(v)\xi = 0$ for only the simple root generating series $T_{ij}(u)$ as given by $T_{i,i+1}(u)$ for $i = 1,\dots,m-1; M+1,\dots,M+n$, and $T_{m,M+1}(u)$.

When $M$ is odd, the same relations (3.1.2) infer that it suffices to show the property $T_{ij}(u)T_{kl}(v)\xi = 0$ for only the simple root generating series $T_{ij}(u)$ as given by $T_{i,i+1}(u)$ for $i = 1,\dots,m-1; M+1,\dots,M+n-1$, and $T_{m,M+1}(u)$, $T_{\widehat{m},M+n+1}(u)$.

*Step 1.* Let us first address the case $1 \le l < k \le m$. Via the relations (3.1.2), it suffices to show $T_{k+1,k}(v)\xi = 0$ for the indices $1 \le k \le m-1$. We use the relations (2.2.8) to evaluate $\big[T_{i,i+1}(u), T_{k+1,k}(v)\big]$ which infers $T_{i,i+1}(u)T_{k+1,k}(v)\xi = 0$ for $i = 1,\dots,k$. Alternatively, calculating $\big[T_{k+1,k}(v), T_{i,i+1}(u)\big]$ will imply $T_{i,i+1}(u)T_{k+1,k}(v)\xi = 0$ for the remaining indices $i = k+1,\dots,m-1; M+1,\dots,M+n$, while one can also find $T_{m,M+1}(u)T_{k+1,k}(v)\xi = 0$ and $T_{\widehat{m},M+n+1}(u)T_{k+1,k}(v)\xi = 0$ via a similar computation.

*Step 2.* For the case $\widehat{m}+1 \le l < k \le M$, it suffices to show $T_{k+1,k}(v)\xi = 0$ for indices $\widehat{m}+1 \le k \le M-1$ via the relations (3.1.2). In particular, we will show

$$\frac{v+\alpha+\kappa-\bar{k}+1}{v+\kappa-m} T_{k+1,k}(v)\xi = -\frac{v+\kappa-\bar{k}+1}{v+\alpha+\kappa-m} T_{\bar{k},\bar{k}-1}(v+\kappa-\bar{k}+1)\xi,$$

which implies the result by Step 1. Using the defining relations (2.2.8) to compute the commutator $\big[T_{k,\bar{k}-1}(u), T_{\bar{k}k}(v)\big]$, we yield

$$\begin{aligned} T_{\bar{k}k}(v)T_{k,\bar{k}-1}(u)\xi &= \frac{\kappa(v+\kappa-m)}{(u-v)(u-v-\kappa)(v+\alpha+\kappa-m)} T_{\bar{k},\bar{k}-1}(u)\xi \\ &+ \frac{u+\kappa-m}{(u-v)(u+\alpha+\kappa-m)} T_{\bar{k},\bar{k}-1}(v)\xi + \frac{1}{u-v-\kappa}\sum_{p=1}^{\bar{k}-1} T_{p,\bar{k}-1}(u)T_{\bar{p}k}(v)\xi. \end{aligned}$$

For $1 \le p \le \bar{k}-1$, evaluating $\big[T_{p,\bar{k}-1}(u), T_{\bar{p}k}(v)\big]$ infers the equality

$$\begin{aligned} T_{p,\bar{k}-1}(u)T_{\bar{p}k}(v)\xi &= \delta_{p,\bar{k}-1}\frac{u+\alpha}{u} T_{k+1,k}(v)\xi + \frac{u+\kappa-m}{(u-v)(u+\alpha+\kappa-m)} T_{\bar{k},\bar{k}-1}(v)\xi \\ &- \frac{v+\kappa-m}{(u-v)(v+\alpha+\kappa-m)} T_{\bar{k},\bar{k}-1}(u)\xi - T_{\bar{k}k}(v)T_{k,\bar{k}-1}(u)\xi, \end{aligned}$$

so one yields the relation

$$(u-v-\kappa+\bar{k}-1)T_{\bar{k}k}(v)T_{k,\bar{k}-1}(u)\xi - \frac{(\kappa-\bar{k}+1)(v+\kappa-m)}{(u-v)(v+\alpha+\kappa-m)}T_{\bar{k},\bar{k}-1}(u)\xi$$
$$= \frac{u+\alpha}{u}T_{k+1,k}(v)\xi + \frac{(u+\kappa-m)(u-v-\kappa+\bar{k}-1)}{(u-v)(u+\alpha+\kappa-m)}T_{\bar{k},\bar{k}-1}(v)\xi.$$

Evaluating at $u = v+\kappa-\bar{k}+1$ therefore gives the desired equation.

(iii) Similar to part (ii), when $M$ is even, the relations (3.1.2) infer that it suffices to show the property $T_{ij}(u)T_{kl}^{(r)}\xi = 0$ for only the simple root generating series $T_{ij}(u)$ as given by $T_{i,i+1}(u)$ for $i = 1,\dots,m-1; M+1,\dots,M+n$, and $T_{m,M+1}(u)$.

When $M$ is odd, the relations (3.1.2) infer that it suffices to show the property $T_{ij}(u)T_{kl}^{(r)}\xi = 0$ for only the simple root generating series $T_{ij}(u)$ as given by $T_{i,i+1}(u)$ for $i = 1,\dots,m-1; M+1,\dots,M+n-1$, $T_{m,M+1}(u)$, and $T_{\widehat{m},M+n+1}(u)$.

We assume $1 \le l \le m$ throughout the remainder of the proof. In the following, Steps 1 to 3 address the case $M+1 \le k \le M+N$, while the last step, Step 4, addresses the case $k = \widehat{m}$ when $M$ is odd.

*Step 1.* Assume $M+1 \le k \le M+N$. For $i \in \{1,\dots,m-1\} \setminus \{l\}$, part (ii) of the lemma infers that computing $\left[T_{i,i+1}(u), T_{kl}(v)\right]$ via the defining relations (2.2.8) will imply $T_{i,i+1}(u)T_{kl}(v)\xi = 0$. When $i = M+1,\dots,M+n$ and $\bar{\imath} \ne k$, using the defining relations again to evaluate $\left[T_{kl}(v), T_{i,i+1}(u)\right]$, along with the fact $T_{k,i+1}(v)\xi = \delta_{k,i+1}\xi$ by part (i) of the lemma, will yield

$$T_{i,i+1}(u)T_{kl}(v)\xi = \frac{1}{v-u}\delta_{k,i+1}\big(T_{il}(v) - T_{il}(u)\big)\xi. \tag{3.2.18}$$

In a similar way, we also find $T_{m,M+1}(u)T_{kl}(v)\xi = (v-u)^{-1}\delta_{k,M+1}\big(T_{ml}(v) - T_{ml}(u)\big)\xi$ and $T_{\widehat{m},M+n+1}(u)T_{kl}(v)\xi = (v-u)^{-1}\delta_{k,M+n+1}\big(T_{\widehat{m}l}(v) - T_{\widehat{m}l}(u)\big)\xi$.

*Step 2.* We shall first prove the assertion for $M+1 \le k \le M+n$ via reverse induction on $1 \le l \le m$. Hence, assume $l = m$ and we will now proceed by induction on $M+1 \le k \le M+n$. For the base case $k = M+1$, Step 1 infers that it only remains to evaluate $T_{m,M+1}(u)T_{M+1,m}(v)\xi$. However, $T_{m,M+1}(u)T_{M+1,m}(v)\xi = (\beta-\alpha)u^{-1}v^{-1}\xi$ in this case, as desired.

Suppose now the induction hypothesis holds for the index $k$. We again see via Step 1 that it only remains to check $T_{k,k+1}(u)T_{k+1,m}(v)\xi$. In this case, we use (3.2.18) to compute $T_{k,k+1}(u)T_{k+1,m}(v)\xi = -u^{-1}v^{-1}T_{km}^{(1)}\xi$ by induction hypothesis.

Now assume the reverse induction hypothesis holds for $1 \leq l+1 \leq m$. Using the defining relations to compute $\left[T_{l,l+1}(u), T_{kl}(v)\right]$, the reverse induction hypothesis will imply $T_{l,l+1}(u)T_{kl}(v)\xi = -u^{-1}v^{-1}T_{k,l+1}^{(1)}\xi$. Furthermore, part (ii) of the lemma and Step 1 infers $T_{m,M+1}(u)T_{kl}(v)\xi = 0$ since $l \leq m-1$. Similar to the case $l = m$, we now proceed by induction on $M+1 \leq k \leq M+n$, where we note the base case $k = M+1$ is automatically satisfied. Supposing the induction hypothesis holds for the index $k$, all that remains to check is the element $T_{k,k+1}(u)T_{k+1,l}(v)\xi$, but (3.2.18) infers $T_{k,k+1}(u)T_{k+1,l}(v)\xi = -u^{-1}v^{-1}T_{kl}^{(1)}\xi$ by induction hypothesis.

*Step 3.* We now prove the assertion for remaining indices $M+n+1 \leq k \leq M+N$. Of course, when $\bar{\imath} = k$ then (3.2.18) does not hold, so we shall derive the suitable relation now in this case. Given $M+1 \leq i \leq M+n-1$, and $i = M+n$ when $M$ is even, we use the defining relations (2.2.8) to compute $\left[T_{\bar{\imath}l}(v), T_{i,i+1}(u)\right]$ and incorporate the fact that $T_{il}(v)\xi = v^{-1}T_{il}^{(1)}\xi$ by Step 2 to obtain the formula

$$T_{i,i+1}(u)T_{\bar{\imath}l}(v)\xi = -\frac{\delta_{i,M+n}}{uv}T_{il}^{(1)}\xi - \frac{\theta_{i+1}}{v-u-\kappa}T_{\overline{i+1},l}(v)\xi + \frac{1}{v-u-\kappa}T_{ll}(v)T_{\bar{l},i+1}(u)\xi + \frac{1}{v-u-\kappa}\sum_{p\in\{1,\dots,m\}\setminus\{l\}} T_{pl}(v)T_{\bar{p},i+1}(u)\xi,$$

noting the use of part (i) to simplify the sum. Again, we note that in the above formula and for the proceeding computations, we assume the index $i = M+n$ occurs only when $M$ is even; that is, we implicitly suppose $\delta_{i,M+n} = 0$ when $M$ is odd.

For $p \in \{1,\dots,m\} \setminus \{l\}$, evaluating $\left[T_{pl}(v), T_{\bar{p},i+1}(u)\right]$ and using part (ii) of the lemma infers $T_{pl}(v)T_{\bar{p},i+1}(u)\xi = -T_{i,i+1}(u)T_{\bar{\imath}l}(v)\xi - \delta_{i,M+n}u^{-1}v^{-1}T_{il}^{(1)}\xi$; hence,

$$T_{i,i+1}(u)T_{\bar{\imath}l}(v)\xi = -\frac{\delta_{i,M+n}}{uv}T_{il}^{(1)}\xi - \frac{\theta_{i+1}}{v-u-\kappa+m-1}T_{\overline{i+1},l}(v)\xi + \frac{1}{v-u-\kappa+m-1}T_{ll}(v)T_{\bar{l},i+1}(u)\xi.$$

Via the defining relations, computing $\left[T_{ll}(v), T_{\bar{l},i+1}(u)\right]$ provides the equation

$$\frac{v-u-\kappa+1}{v-u-\kappa}T_{ll}(v)T_{\bar{l},i+1}(u)\xi = \frac{v+\alpha}{v}T_{\bar{l},i+1}(u)\xi + \frac{\theta_{i+1}}{v-u-\kappa}T_{\overline{i+1},l}(v)\xi - \frac{1}{v-u-\kappa}\sum_{p\in\{1,\dots,m\}\setminus\{l\}} T_{pl}(v)T_{\bar{p},i+1}(u)\xi.$$

Similar to before, evaluating $[T_{pl}(v), T_{\bar{p},i+1}(u)]$ for indices $p \in \{1,\dots,m\} \setminus \{l\}$ implies $T_{pl}(v)T_{\bar{p},i+1}(u)\xi = T_{ll}(v)T_{\bar{l},i+1}(u)\xi - (1+\alpha v^{-1})T_{\bar{l},i+1}(u)\xi$, so we obtain

$$T_{ll}(v)T_{\bar{l},i+1}(u)\xi = \frac{(v-u-\kappa+m-1)(v+\alpha)}{v(v-u-\kappa+m)}T_{\bar{l},i+1}(u)\xi + \frac{\theta_{i+1}}{v-u-\kappa+m}T_{\overline{i+1},l}(v)\xi.$$

Taking the residue at $v = u+\kappa-m$ gives

$$T_{\bar{l},i+1}(u)\xi = \theta_{i+1}\frac{u+\kappa-m}{u+\alpha+\kappa-m}T_{\overline{i+1},l}(u+\kappa-m)\xi,$$

and so we finally deduce

$$\begin{aligned} T_{i,i+1}(u)T_{\bar{i}l}(v)\xi = -\frac{\delta_{i,M+n}}{uv}T^{(1)}_{il}\xi - \frac{\theta_{i+1}}{v-u-\kappa+m}T_{\overline{i+1},l}(v)\xi \\ + \frac{\theta_{i+1}(u+\kappa-m)(v+\alpha)}{v(v-u-\kappa+m)(u+\alpha+\kappa-m)}T_{\overline{i+1},l}(u+\kappa-m)\xi. \end{aligned} \tag{3.2.19}$$

We now proceed via reverse induction on $1 \le l \le m$. Similar to before, we assume $l = m$ and now proceed via induction on $M+n+1 \le k \le M+N$. For the base case $k = M+n+1$, Step 1 infers that it only remains to check the elements $T_{\hat{m}k}(u)T_{kl}(v)\xi$ when $M$ is odd and $T_{k-1,k}(u)T_{kl}(v)\xi$ when $M$ is even. However, when $M$ is odd, Step 4 below will show

$$T_{k-1,k}(u)T_{kl}(v)\xi = -\frac{1}{uv}T^{(1)}_{\hat{m}l}\xi,$$

while if $M$ is even we utilize Step 2 and (3.2.19) to deduce

$$T_{k-1,k}(u)T_{kl}(v)\xi = -\left(\frac{1}{uv} + \frac{1}{v(u+\alpha+\kappa-m)}\right)T^{(1)}_{k-1,l}\xi. \tag{3.2.20}$$

Assuming the induction hypothesis holds for the index $k$, it only remains to check the element $T_{\overline{k+1},\bar{k}}(u)T_{k+1,m}(v)\xi$ by Step 1. However, via (3.2.19) and the induction hypothesis, we conclude

$$T_{\overline{k+1},\bar{k}}(u)T_{k+1,l}(v)\xi = \frac{1}{v(u+\alpha+\kappa-m)}T^{(1)}_{kl}\xi. \tag{3.2.21}$$

Now assume the reverse induction hypothesis holds for $1 \le l+1 \le m$. Using the defining relations to compute $[T_{l,l+1}(u), T_{kl}(v)]$, the reverse induction hypothesis will imply $T_{l,l+1}(u)T_{kl}(v)\xi = -u^{-1}v^{-1}T^{(1)}_{k,l+1}\xi$. Similar to the case $l = m$, we now proceed by induction on $M+n+1 \le k \le M+N$. For the base case $k = M+n+1$, all that is left to be checked is $T_{\widehat{m}k}(u)T_{kl}(v)\xi$ when $M$ is odd and $T_{k-1,k}(u)T_{kl}(v)\xi$ when $M$ is even, but we may similarly deduce $T_{\widehat{m}k}(u)T_{kl}(v)\xi = -u^{-1}v^{-1}T^{(1)}_{\widehat{m}l}\xi$ in the first case by Step 4 of the lemma and conclude the relation (3.2.20) in the second case from Step 2 and (3.2.19). Supposing the induction hypothesis holds for the index $k$, all that remains to check is the element $T_{\overline{k+1},\bar{k}}(u)T_{k+1,m}(v)\xi$, but the induction hypothesis and (3.2.19) will imply (3.2.21).

*Step 4.* Assume $M$ is odd and $k = \widehat{m}$. We shall first prove $T_{\widehat{m}l}(v)\xi = T^{(1)}_{\widehat{m}l}\xi$ for such $1 \le l \le m$. For indices $i \in \{1, \dots, m-1\} \setminus \{l\}$, we use the defining relations to evaluate $\left[T_{i,i+1}(u), T_{\widehat{m}l}(v)\right]$ and use part (ii) to infer $T_{i,i+1}(u)T_{\widehat{m}l}(v)\xi = 0$. For indices $i = M+1, \dots, M+n-1$, we calculate the commutator $\left[T_{\widehat{m}l}(v), T_{i,i+1}(u)\right]$ and use part (i) of the lemma to conclude $T_{i,i+1}(u)T_{\widehat{m}m}(v)\xi = 0$. We can similarly reason $T_{m,M+1}(u)T_{\widehat{m}l}(v)\xi = 0$, but we observe that computing $\left[T_{\widehat{m}l}(v), T_{\widehat{m},M+n+1}(u)\right]$ gives

$$T_{\widehat{m},M+n+1}(u)T_{\widehat{m}l}(v)\xi = \frac{1}{v(v-u-\kappa)}T^{(1)}_{M+n,l}\xi + \frac{1}{v-u-\kappa}\sum_{p=1}^{m} T_{pl}(v)T_{\bar{p},M+n+1}(u)\xi$$

by Step 2. Evaluating the commutator $\left[T_{pl}(v), T_{\bar{p},M+n+1}(u)\right]$ for indices $1 \le p \le m$ will infer the equality $T_{pl}(v)T_{\bar{p},M+n+1}(u)\xi = \delta_{pl}(1+\alpha v^{-1})T_{\bar{l},M+n+1}(u)\xi - T_{\widehat{m},M+n+1}(u)T_{\widehat{m}l}(v)\xi$, so we deduce

$$T_{\widehat{m},M+n+1}(u)T_{\widehat{m}l}(v)\xi = \frac{1}{v(v-u-\kappa+m)}\Big(T^{(1)}_{M+n,l}\xi + (v+\alpha)T_{\bar{l},M+n+1}(u)\xi\Big).$$

Taking the residue at $v = u+\kappa-m$ therefore implies

$$T_{\bar{l},M+n+1}(u)\xi = -\frac{1}{u+\alpha+\kappa-m}T^{(1)}_{M+n,l}\xi,$$

which in turn infers $T_{\widehat{m},M+n+1}(u)T_{\widehat{m}l}(v)\xi = -v^{-1}T^{(1)}_{M+n,l}\xi$.

We now proceed by reverse induction on $1 \le l \le m$, with the above calculations establishing the base case $l = m$. Assuming the induction hypothesis holds down to the index $l+1$, the only vector remaining to check is $T_{l,l+1}(u)T_{\widehat{m}l}(v)\xi$. To this end, we evaluate $\left[T_{l,l+1}(u), T_{\widehat{m}l}(v)\right]$ and use the induction hypothesis to obtain

$$T_{l,l+1}(u)T_{\widehat{m}l}(v)\xi = -\frac{1}{uv}T^{(1)}_{\widehat{m},l+1}\xi,$$

finishing the proof. □

**Lemma 3.2.13.** *Let $\xi$ denote the highest weight vector of $L(\lambda(u);\alpha,0)$ as in Proposition 3.2.11. For indices $\widehat{m}+1 \le k \le M$, $1 \le l \le m$ and integers $r \in \mathbb{Z}^+$, there are uniquely determined constants $\gamma_{kl}^{(r)} \in \mathbb{C}$ such that*

$$T_{kl}^{(r)}\xi = \gamma_{kl}^{(r)} \sum_{p=M+1}^{M+n} T_{\bar{p}\bar{k}}^{(1)} T_{pl}^{(1)}\xi, \quad \textit{where} \quad \gamma_{kl}^{(1)} = 0. \tag{3.2.22}$$

*When $M$ is even, the constants are determined by $\sum_{r=1}^{\infty} \gamma_{kl}^{(r)} v^{-r} = \frac{1}{2}v^{-1}(v+\alpha+\kappa-m)^{-1}$.*

*Proof.* We will first show $T_{kl}^{(1)}\xi = 0$ for indices $\widehat{m}+1 \le k \le M$, $1 \le l \le m$. By relation (3.1.1), $T_{k\bar{k}}^{(1)} = 0$ for indices $\widehat{m}+1 \le k \le M$, so we may assume $\bar{k} \ne l$. Moreover, the same relation infers that it suffices to prove $T_{kl}^{(1)}\xi = 0$ for $1 \le l \le m-1$ and $\widehat{m}+2 \le k < \bar{l}$.

Under the embedding (2.4.7), $L(\lambda(u);\alpha,0)$ is given a $\mathfrak{U}(\mathfrak{osp}_{M|N})$-module structure such that $F_{kl}\xi = T_{kl}^{(1)}\xi$ for indices $\widehat{m}+1 \le k \le M$, $1 \le l \le m$, so it therefore suffices to show $T_{kl}^{(1)}\xi$ is a singular vector under such representation.

When $M$ is even, the relations (3.1.3) infer that it suffices to show the property $F_{ij}T_{kl}^{(1)}\xi = 0$ for only the simple root vectors $F_{ij}$ as given by $F_{i,i+1}$ for $i = 1,\dots,m-1$; $M+1,\dots,M+n$, and $F_{m,M+1}$.

Similarly, when $M$ is odd, the relations (3.1.3) infer that it suffices to show the property $F_{ij}T_{kl}^{(1)}\xi = 0$ for only the simple root vectors $F_{ij}$ as given by $F_{i,i+1}$ for $i = 1,\dots,m-1$; $M+1,\dots,M+n-1$, $F_{m,M+1}$, and $F_{\widehat{m},M+n+1}$.

By relations (3.1.3), we see $F_{i,i+1}T_{kl}^{(1)}\xi = [F_{i,i+1}, T_{kl}^{(1)}]\xi = 0$ for $i = M+1,\dots,M+n$, including $F_{m,M+1}T_{kl}^{(1)}\xi = [F_{m,M+1}, T_{kl}^{(1)}]\xi = -\delta_{ml}T_{k,M+1}^{(1)}\xi + \delta_{\widehat{m}+1,k}T_{M+N,l}^{(1)}\xi$, and when $M$ is odd: $F_{\widehat{m},M+n+1}T_{kl}^{(1)}\xi = [F_{\widehat{m},M+n+1}, T_{kl}^{(1)}]\xi = 0$. However, for $i = 1,\dots,m-1$ we compute

$$F_{i,i+1}T_{kl}^{(1)}\xi = [F_{i,i+1}, T_{kl}^{(1)}]\xi = -\delta_{il}T_{k,l+1}^{(1)}\xi - \delta_{\bar{i}k}T_{k-1,l}^{(1)}\xi.$$

We shall first prove $T_{kl}^{(1)}\xi = 0$ for indices $1 \le l \le m-2$ and $\widehat{m}+2 \le k < \bar{l}$ by reverse induction on $1 \le l \le m-2$. When $l = m-2$, it must be that $k = \widehat{m}+2$, so the above discussion infers that we only need to check $F_{l,l+1}T_{kl}^{(1)}\xi = -T_{k,l+1}^{(1)}\xi = T_{k\bar{k}}^{(1)}\xi$ which is zero since $T_{k\bar{k}}^{(1)} = 0$. Assuming the induction hypothesis holds for $1 \le l+1 \le m-2$, we similarly only need to compute the relation $F_{l,l+1}T_{kl}^{(1)}\xi = -T_{k,l+1}^{(1)}\xi$, but such is zero by induction for $\widehat{m}+1 \le k < \bar{l}-1$, whereas $T_{\bar{l}-1,l+1}^{(1)}\xi = T_{\overline{l+1},l+1}^{(1)}\xi = 0$.

For $1 \le l \le m-2$, we observe $\big[T^{(1)}_{\widehat{m}+1,k}, T^{(1)}_{kl}\big] = T^{(1)}_{\widehat{m}+1,l}$ for all $\widehat{m}+2 \le k < \bar{l}$; hence, $T^{(1)}_{\widehat{m}+1,l}\xi = 0$ for $1 \le l \le m-2$ by the argument above. Similarly, for such indices we see $\big[T^{(1)}_{\widehat{m}+1,l}, T^{(1)}_{l,m-1}\big] = T^{(1)}_{\widehat{m}+1,m-1}$, so $T^{(1)}_{\widehat{m}+1,m-1}\xi = 0$ as well. We now complete the remainder of the proof in two steps:

*Step 1.* By computing $\big[T_{\bar{k}l}(v), T_{k\bar{k}}(u)\big]$, we use part (ii) of Lemma 3.2.12 to provide

$$\frac{v-u-\kappa+1}{v-u-\kappa}T_{\bar{k}l}(v)T_{k\bar{k}}(u)\xi = \frac{(\delta_{\bar{k}l}(v-u)-1)(v+\alpha)}{v(v-u)}T_{kl}(u)\xi - \frac{\kappa(u+\alpha)}{u(v-u)(v-u-\kappa)}T_{kl}(v)\xi \\ - \frac{1-\delta_{\bar{k}l}}{v-u-\kappa}T_{ll}(v)T_{\bar{l}\bar{k}}(u)\xi - \frac{1}{v-u-\kappa}\sum_{\substack{p\in\{1,\dots,m\}\setminus\{\bar{k},l\},\\ M+1\le p\le M+N}} \theta_p T_{pl}(v)T_{\bar{p}\bar{k}}(u)\xi.$$

For indices $p \in \{1,\dots,m\} \setminus \{\bar{k}, l\}$, evaluating $\big[T_{pl}(v), T_{\bar{p}\bar{k}}(u)\big]$ and using part (ii) Lemma 3.2.12 will infer the equality

$$T_{pl}(v)T_{\bar{p}\bar{k}}(u)\xi = T_{\bar{k}l}(v)T_{k\bar{k}}(u)\xi - \frac{u+\alpha}{u(v-u)}T_{kl}(v)\xi - \frac{(\delta_{\bar{k}l}(v-u)-1)(v+\alpha)}{v(v-u)}T_{kl}(u)\xi,$$

and similarly,

$$(1-\delta_{\bar{k}l})T_{ll}(v)T_{\bar{l}\bar{k}}(u)\xi = \frac{(1-\delta_{\bar{k}l})(v+\alpha)}{v}T_{\bar{l}\bar{k}}(u)\xi + (1-\delta_{\bar{k}l})T_{\bar{k}l}(v)T_{k\bar{k}}(u)\xi \\ - \frac{(1-\delta_{\bar{k}l})(u+\alpha)}{u(v-u)}T_{kl}(v)\xi + \frac{(1-\delta_{\bar{k}l})(\delta_{\bar{k}l}(v-u)-1)(v+\alpha)}{v(v-u)}T_{kl}(u)\xi.$$

Hence, we find that $T_{\bar{k}l}(v)T_{k\bar{k}}(u)\xi$ is equal to the expression

$$\frac{(\delta_{\bar{k}l}(v-u)-1)(v-u-\kappa+m-1)(v+\alpha)}{v(v-u)(v-u-\kappa+m)}T_{kl}(u)\xi - \frac{(\kappa-m+1)(u+\alpha)}{u(v-u)(v-u-\kappa+m)}T_{kl}(v)\xi \\ - \frac{(1-\delta_{\bar{k}l})(v+\alpha)}{v(v-u-\kappa+m)}T_{\bar{l}\bar{k}}(u)\xi - \frac{1}{u(v-u-\kappa+m)}\sum_{p=M+1}^{M+N}\theta_p T_{pl}(v)T^{(1)}_{\bar{p}\bar{k}}\xi,$$

where we used part (iii) of Lemma 3.2.12 for the terms occurring in the sum. Using the previous lemma again along with the formula $\big[T^{(1)}_{\bar{p}\bar{k}}, T_{pl}(v)\big] = -(-1)^{[p]}\theta_{\bar{p}}T_{kl}(v)$, taking the residue $u = v-\kappa+m$ therefore gives

$$\frac{(\delta_{\bar{k}l}(\kappa-m)-1)(v+\alpha)}{(\kappa-m)v}T_{kl}(v-\kappa+m)\xi + \frac{(\kappa-m+1)(v+\alpha-\kappa+m)-(\kappa-m)N}{(\kappa-m)(v-\kappa+m)}T_{kl}(v)\xi \\ = -\frac{(1-\delta_{\bar{k}l})(v+\alpha)}{v}T_{\bar{l}\bar{k}}(v-\kappa+m)\xi + \frac{1}{v(v-\kappa+m)}\sum_{p=M+1}^{M+N}\theta_p T^{(1)}_{\bar{p}\bar{k}}T^{(1)}_{pl}\xi.$$

When $\bar{k}=l$, we realize that the coefficient of $v^{-r}$ on the left hand side of such equality is given by $2T^{(r)}_{k\bar{k}}\xi \mod \mathbf{E}_{r-2}\xi$, where $\mathbf{E}=\{\mathbf{E}_r\}_{r\in\mathbb{N}}$ is the filtration defined by (2.2.21).

Therefore, the above equation uniquely determines the action of $T^{(r)}_{k\bar{k}}\xi$ by an inductive argument on $r\in\mathbb{Z}^+$. In particular, since $T^{(1)}_{k\bar{k}}=0$, then $\left[T^{(1)}_{\bar{p}\bar{k}},T^{(1)}_{p\bar{k}}\right]=0$ which gives the final form. Furthermore, when $M$ is even we observe

$$T_{k\bar{k}}(v)\xi=\frac{1}{2v(v+\alpha+\kappa-m)}\sum_{p=M+1}^{M+N}\theta_p T^{(1)}_{\bar{p}\bar{k}}T^{(1)}_{p\bar{k}}\xi$$

satisfies the prior equation.

*Step 2.* Let us now assume $\bar{k}\neq l$ for the remainder of the proof. By equation (2.2.18), we have $\sum_{p=1}^{M+N}T^{t}_{kp}(v-\kappa+m)T_{pl}(v-2\kappa+m)=0$. Therefore, by Lemma 3.2.12 and relations (3.1.2), we deduce

$$\begin{aligned}\frac{v+\alpha-2\kappa-N+m}{v-2\kappa+m}T_{\bar{l}\bar{k}}(v-\kappa+m)\xi=&-\sum_{p=\widehat{m}+1}^{M}T_{\bar{p}\bar{k}}(v-\kappa+m)T_{pl}(v-2\kappa+m)\xi\\&-\frac{1}{(v-\kappa+m)(v-2\kappa+m)}\sum_{p=M+1}^{M+N}\theta_p T^{(1)}_{pl}T^{(1)}_{\bar{p}\bar{k}}\xi\end{aligned}$$

Hence, by combining the above equation with the one in Step 1 and using the relation $\left[T^{(1)}_{pl},T^{(1)}_{\bar{p}\bar{k}}\right]\xi=\theta_p T^{(1)}_{\bar{l}\bar{k}}\xi=0$ for $M+1\leq p\leq M+N$, we yield

$$\begin{aligned}&\frac{(\kappa-m+1)(v+\alpha-\kappa+m)-(\kappa-m)N}{(\kappa-m)(v-\kappa+m)}T_{kl}(v)\xi-\frac{v+\alpha}{(\kappa-m)v}T_{kl}(v-\kappa+m)\xi\\&\quad-\frac{(v-2\kappa+m)(v+\alpha)}{v(v+\alpha-2\kappa-N+m)}\sum_{p=\widehat{m}+1}^{M}T_{\bar{p}\bar{k}}(v-\kappa+m)T_{pl}(v-2\kappa+m)\xi\\&\qquad=\frac{-2\kappa-N+m}{v(v-\kappa+m)(v+\alpha-2\kappa-N+m)}\sum_{p=M+1}^{M+N}\theta_p T^{(1)}_{\bar{p}\bar{k}}T^{(1)}_{pl}\xi.\end{aligned}$$

The coefficient of $v^{-r}$ on the left hand side of the above equation can be written as a sum $\sum_{s=2}^{r-1}c^{(s)}_{kl}T^{(s)}_{kl}$ for certain constants $c^{(s)}_{kl}\in\mathbb{C}$ where $c^{(r-1)}_{kl}\neq 0$.

Hence, the above equation uniquely determines the action of $T^{(r)}_{kl}\xi$ for $r\geq 2$ inductively. Since $\left[T^{(1)}_{\bar{p}\bar{k}},T^{(1)}_{pl}\right]\xi=-\theta_p T^{(1)}_{kl}\xi=0$ for the indices $M+1\leq p\leq M+N$, we get the final form. Furthermore, when $M$ is even we observe

$$T_{kl}(v)\xi = \frac{1}{2v(v+\alpha+\kappa-m)} \sum_{p=M+1}^{M+N} \theta_p T^{(1)}_{\bar{p}\bar{k}} T^{(1)}_{pl} \xi$$

satisfies the prior equation, or equivalently the equation as in Step 1. □

We are now in position to prove the proposition.

*Proof of Proposition 3.2.11.* We shall demarcate the proof in two steps according to the parity of $M$.

*Step 1.* Assume $M$ is even. Considering the subset $C \subset (\mathbb{Z}^+)^2$ defined by the collection $\{(k,l) \mid M+1 \leq k \leq M+N,\ 1 \leq l \leq m\}$, if '$\preceq$' is any total ordering on the set $\{T^{(1)}_{kl} \mid (k,l) \in C\}$ we claim that

$$L(\lambda(u);\alpha,0) = \mathrm{span}_{\mathbb{C}} \{T^{(1)}_{k_1l_1} \cdots T^{(1)}_{k_rl_r}\xi \mid (k_i,l_i) \in C,\ T^{(1)}_{k_il_i} \preceq T^{(1)}_{k_{i+1}l_{i+1}},\ r \in \mathbb{Z}^+\}.$$

Let $\mathcal{V}$ denote the span on the right hand side of the above equation. By the irreducibility of $L(\lambda(u);\alpha,0)$ and the PBW Theorem for $\mathrm{X}(\mathfrak{osp}_{M|N})$ (see Corollary 2.4.5), it therefore suffices to show that $\mathcal{V}$ is invariant under the action of $T_{ij}(u)$ for $(i,j) \in \mathcal{B}_{M|N}$ as in (2.3.15).

Via the relations (3.1.2), we find $[T_{kl}(u), T^{(1)}_{k'l'}] = -\delta_{\bar{k}k'}\theta_k T_{\bar{l}'l}(u)$ for $(k,l),(k',l') \in C$. Furthermore, since $[T_{ij}(u), T^{(1)}_{kl}] = 0$ for indices $m+1 \leq i \leq M$, $1 \leq j \leq m$ and $(k,l) \in C$, the action of $T_{kl}(u)$ on each monomial $T^{(1)}_{k_1l_1} \cdots T^{(1)}_{k_rl_r}\xi$ in $\mathcal{V}$ is given by

$$(-1)^r T^{(1)}_{k_1l_1} \cdots T^{(1)}_{k_rl_r} T_{kl}(u)\xi + \sum_{a=1}^{r} (-1)^{a-1} T^{(1)}_{k_1l_1} \cdots [T_{kl}(u), T^{(1)}_{k_al_a}] \cdots T^{(1)}_{k_rl_r}\xi,$$

which is equal to

$$(-1)^r T^{(1)}_{k_1l_1} \cdots T^{(1)}_{k_rl_r} T_{kl}(u)\xi + \theta_k \sum_{a=1}^{r} (-1)^a \delta_{\bar{k}k_a} T^{(1)}_{k_1l_1} \cdots \widehat{T}^{(1)}_{k_al_a} \cdots T^{(1)}_{k_rl_r} T_{\bar{l}_al}(u)\xi,$$

where $\widehat{T}^{(1)}_{k_al_a}$ denotes the omitted term. Hence, by part (iii) of Lemma 3.2.12 and Lemma 3.2.13, we can conclude $T_{ij}(u)\mathcal{V} \subseteq \mathcal{V}$ for indices $M+1 \leq i \leq M+N$, $1 \leq j \leq m$. In fact, we observed such is also true for $m+1 \leq i \leq M$, $1 \leq j \leq m$.

We shall now determine $T_{ij}(v)\mathcal{V} \subseteq \mathcal{V}$ for $m+1 \le i \le M$ and $M+1 \le j \le M+N$, starting with the action of $T_{ij}(v)$ on the vector $\xi$. Supposing $1 \le b \le m$, parts (i) and (ii) of Lemma 3.2.12 imply that computing the commutator $\big[T_{b\bar{\imath}}(u), T_{\bar{b}j}(v)\big]$ gives the expression

$$T_{b\bar{\imath}}(u)T_{\bar{b}j}(v)\xi = \delta_{\bar{\imath}b}\frac{u+\alpha}{u}T_{ij}(v)\xi - \frac{1}{u(u-v-\kappa)}T^{(1)}_{\bar{\jmath}\bar{\imath}}\xi - \frac{1}{u-v-\kappa}\sum_{p=1}^{m} T_{p\bar{\imath}}(u)T_{\bar{p}j}(v)\xi.$$

Evaluating the commutator $\big[T_{p\bar{\imath}}(u), T_{\bar{p}j}(v)\big]$ for indices $1 \le p \le m$ will infer the equality $T_{p\bar{\imath}}(u)T_{\bar{p}j}(v)\xi = (\delta_{\bar{\imath}p} - \delta_{\bar{\imath}b})(1+\alpha u^{-1})T_{ij}(v)\xi + T_{b\bar{\imath}}(u)T_{\bar{b}j}(v)\xi$, so we deduce

$$T_{b\bar{\imath}}(u)T_{\bar{b}j}(v)\xi = \frac{1}{u(u-v-\kappa+m)}\Big((u+\alpha)(\delta_{\bar{\imath}b}(u-v-\kappa+m)-1)T_{ij}(v)\xi - T^{(1)}_{\bar{\jmath}\bar{\imath}}\xi\Big).$$

Taking the residue at $u = v+\kappa-m$ therefore implies

$$T_{ij}(v)\xi = -\frac{1}{v+\alpha+\kappa-m}T^{(1)}_{\bar{\jmath}\bar{\imath}}\xi.$$

Furthermore, we observe $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{jk}T_{il}(u)$, so the above discussion establishes the fact $T_{ij}(v)\mathcal{V} \subseteq \mathcal{V}$.

When $M+1 \le i,j \le M+N$, we compute $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{jk}T_{il}(u) + \delta_{\bar{\imath}k}\theta_i T_{\bar{l}j}(u)$ for $(k,l) \in C$; thus, by part (i) of Lemma 3.2.12 and the prior discussion, we establish $T_{ij}(u)\mathcal{V} \subseteq \mathcal{V}$ for such indices $M+1 \le i,j \le M+N$.

The rest of the proof proceeds by systematically showing $T_{ij}(u)\mathcal{V} \subseteq \mathcal{V}$ for the remaining indices $(i,j) \in \mathcal{B}_{M|N}$ with similar argument, so we shall only outline how to yield the remaining desired indices.

For indices $m+1 \le i,j \le M$, we find $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{\bar{\jmath}l}\theta_k T_{i\bar{k}}(u)$ for $(k,l) \in C$; thus, when $M+1 \le i \le M+N$ and $m+1 \le j \le M$, the commutator $\big[T_{ij}(u), T^{(1)}_{kl}\big]$ is given by $-\delta_{\bar{\imath}k}\theta_i T_{\bar{l}j}(u) - \delta_{\bar{\jmath}l}\theta_k T_{i\bar{k}}(u)$, which shows $T_{ij}(u)\mathcal{V} \subseteq \mathcal{V}$.

Given $1 \le i,j \le m$, we find $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{il}T_{kj}(u)$ for $(k,l) \in C$, which proves $T_{ij}(u)\mathcal{V} \subseteq \mathcal{V}$.

For indices $1 \le i \le m$ and $M+1 \le j \le M+N$, one computes $\big[T_{ij}(u), T^{(1)}_{kl}\big]$ to be $\delta_{il}T_{kj}(u) - \delta_{jk}T_{il}(u)$ for $(k,l) \in C$; hence, when we have $1 \le i \le m$, $m+1 \le j \le M$, then $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{il}T_{kj}(u) - \delta_{\bar{\jmath}l}\theta_k T_{i\bar{k}}(u)$, which therefore shows $T_{ij}(u)\mathcal{V} \subseteq \mathcal{V}$ for the remaining indices. In particular, we establish the equality $L(\lambda(u);\alpha,0)=\mathcal{V}$, so $\dim L(\lambda(u);\alpha,0)\leq 2^{mN}$.

Since $\mathcal{Z}_1\xi=0$, the embedding (2.4.7) equips $L(\lambda(u);\alpha,0)$ with a $\mathfrak{U}(\mathfrak{osp}_{M|N})$-module structure determined by $F_{ij}\xi=(-1)^{[i]}T_{ij}^{(1)}\xi$ for $1\leq i,j\leq M+N$. As established in the proof, we have the equality $\mathfrak{U}(\mathfrak{osp}_{M|N})\xi=L(\lambda(u);\alpha,0)$, so the quotient of $L(\lambda(u);\alpha,0)$ by its maximal $\mathfrak{osp}_{M|N}$-submodule will be isomorphic to the irreducible representation $V(\lambda^{(1)})$ of $\mathfrak{osp}_{M|N}$ with highest weight

$$\lambda^{(1)}=(\lambda_1^{(1)},\dots,\lambda_m^{(1)},\lambda_{M+1}^{(1)},\dots,\lambda_{M+n}^{(1)})=(\underbrace{\alpha,\dots,\alpha}_{m},\underbrace{0,\dots,0}_{n}).$$

Since $V(\lambda^{(1)})$ is finite-dimensional, the necessary conditions on the highest weight $\lambda^{(1)}$ forces the requirement $\alpha\in\frac{1}{2}\mathbb{Z}^+$.

Conversely, suppose $\alpha\in\frac{1}{2}\mathbb{Z}^+$ such that the highest weight $\mu=(\alpha,\dots,\alpha,0,\dots,0)$ of the above form makes the irreducible $\mathfrak{osp}_{M|N}$-module $V(\mu)$ is finite-dimensional. If $\zeta$ denotes the highest weight vector of $V(\mu)$, we can compute

$$V(\mu)=\operatorname{span}_{\mathbb{C}}\left\{F_{k_1l_1}\cdots F_{k_rl_r}\zeta \mid (k_i,l_i)\in C,\ F_{k_il_i}\preceq F_{k_{i+1}l_{i+1}},\ r\in\mathbb{Z}^+\right\},$$

where $C$ and '$\preceq$' are defined similar to before. In particular, we deduce there is an isomorphism $L(\lambda(u);\alpha,0)\cong V(\mu)$ of $\mathfrak{osp}_{M|N}$-modules.

*Step 2.* Assume $M$ is odd. Recalling the subset $C\subset(\mathbb{Z}^+)^2$ from Step 1, if we suppose '$\preceq$' is any total ordering on the set $\{T_{kl}^{(1)},T_{\widehat{m}b}^{(1)}\mid(k,l)\in C,\ 1\leq b\leq m\}$ such that $T_{\widehat{m}b}^{(1)}\preceq T_{kl}^{(1)}$ for any indices $1\leq b\leq m$ and $(k,l)\in C$, we claim that the module $L(\lambda(u);\alpha,0)$ is equal to the following span of ordered monomials:

$$\mathcal{W}:=\operatorname{span}_{\mathbb{C}}\left\{\left(T_{\widehat{m}b_1}^{(1)}\cdots T_{\widehat{m}b_s}^{(1)}\right)\left(T_{k_1l_1}^{(1)}\cdots T_{k_rl_r}^{(1)}\right)\xi\mid 1\leq b_j\leq m,\ (k_i,l_i)\in C,\ s,r\in\mathbb{Z}^+\right\},$$

where $T_{\widehat{m}b_j}^{(1)}\preceq T_{\widehat{m}b_{j+1}}^{(1)}$ and $T_{k_il_i}^{(1)}\preceq T_{k_{i+1}l_{i+1}}^{(1)}$ for $1\leq j\leq s-1$, $1\leq i\leq r-1$.

The proof is similar to Step 1, where it suffices to show $\mathcal{W}$ is invariant under the action of $T_{ij}(u)$ for $(i,j)\in\mathcal{B}_{M|N}$ via the irreducibility of $L(\lambda(u);\alpha,0)$ and the PBW Theorem for $\mathrm{X}(\mathfrak{osp}_{M|N})$.

We note that $\left[T_{\widehat{m}b}(u),T_{\widehat{m}b'}^{(1)}\right]=T_{\overline{b'}b}(u)$ for all $1\leq b,b'\leq m$ and $\left[T_{\widehat{m}b}(u),T_{kl}^{(1)}\right]=0$ for $(k,l)\in C$. Since $\left[T_{ij}(u),T_{\widehat{m}b}^{(1)}\right]=0=\left[T_{ij}(u),T_{kl}^{(1)}\right]$ for all $\widehat{m}+1\leq i\leq M$, $1\leq l\leq m$, we can conclude $T_{\widehat{m}b}(u)\mathcal{W} \subseteq \mathcal{W}$ and $T_{ij}(u)\mathcal{W} \subseteq \mathcal{W}$ by part (iii) of Lemma 3.2.12 and Lemma 3.2.13. Similarly, we can use that $\big[T_{kl}(u), T^{(1)}_{\widehat{m}b}\big] = 0$ for $(k,l) \in C$ and the discussion in Step 1 to infer $T_{kl}(u)\mathcal{W} \subseteq \mathcal{W}$ as well.

We shall now determine $T_{i\widehat{m}}(v)\mathcal{W} \subseteq \mathcal{W}$ for $\widehat{m}+1 \leq i \leq M$, starting with the action of $T_{i\widehat{m}}(v)$ on the vector $\xi$. Supposing $1 \leq b \leq m$, Lemma 3.2.12 implies that computing the commutator $\big[T_{b\bar{\imath}}(u), T_{\bar{b}\widehat{m}}(v)\big]$ gives the expression

$$T_{b\bar{\imath}}(u)T_{\bar{b}\widehat{m}}(v)\xi = \delta_{\bar{\imath}b}\frac{u+\alpha}{u}T_{i\widehat{m}}(v)\xi - \frac{1}{u(u-v-\kappa)}T^{(1)}_{\widehat{m}\bar{\imath}}\xi - \frac{1}{u-v-\kappa}\sum_{p=1}^{m} T_{p\bar{\imath}}(u)T_{\bar{p}\widehat{m}}(v)\xi.$$

Evaluating the commutator $\big[T_{p\bar{\imath}}(u), T_{\bar{p}\widehat{m}}(v)\big]$ for indices $1 \leq p \leq m$ will infer the equality $T_{p\bar{\imath}}(u)T_{\bar{p}\widehat{m}}(v)\xi = (\delta_{\bar{\imath}p} - \delta_{\bar{\imath}b})(1 + \alpha u^{-1})T_{i\widehat{m}}(v)\xi + T_{b\bar{\imath}}(u)T_{\bar{b}\widehat{m}}(v)\xi$, so we deduce

$$T_{b\bar{\imath}}(u)T_{\bar{b}\widehat{m}}(v)\xi = \frac{1}{u(u-v-\kappa+m)}\Big((u+\alpha)(\delta_{\bar{\imath}b}(u-v-\kappa+m)-1)T_{i\widehat{m}}(v)\xi - T^{(1)}_{\widehat{m}\bar{\imath}}\xi\Big).$$

Taking the residue at $u = v + \kappa - m$ therefore implies

$$T_{i\widehat{m}}(v)\xi = -\frac{1}{v+\alpha+\kappa-m}T^{(1)}_{\widehat{m}\bar{\imath}}\xi.$$

Furthermore, since $\big[T_{i\widehat{m}}(u), T^{(1)}_{\widehat{m}b}\big] = T_{ib}(u)$ for $1 \leq b \leq m$ and $\big[T_{i\widehat{m}}(u), T^{(1)}_{kl}\big] = 0$ for $(k,l) \in C$, we can establish $T_{i\widehat{m}}(v)\mathcal{W} \subseteq \mathcal{W}$.

When $M+1 \leq i \leq M+N$, we compute $\big[T_{i\widehat{m}}(u), T^{(1)}_{\widehat{m}b}\big] = T_{ib}(u)$ for $1 \leq b \leq m$ and $\big[T_{i\widehat{m}}(u), T^{(1)}_{kl}\big] = \delta_{\bar{\imath}k}\theta_k T_{\bar{l}\widehat{m}}(u)$ for $(k,l) \in C$. By part (i) of Lemma 3.2.12 and the above discussion, we can therefore conclude $T_{i\widehat{m}}(v)\mathcal{W} \subseteq \mathcal{W}$ for the indices $M+1 \leq i \leq M+N$.

Now, for indices $\widehat{m}+1 \leq i \leq M$ and $M+1 \leq j \leq M+N$, we can deploy an identical argument as in Step 1 to determine the formula $T_{ij}(v)\xi = -(v+\alpha+\kappa-m)^{-1}T^{(1)}_{\bar{\jmath}\bar{\imath}}\xi$. In particular, since $\big[T_{ij}(u), T^{(1)}_{\widehat{m}b}\big] = 0$ for $1 \leq b \leq m$ and $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{jk}T_{il}(u)$ for $(k,l) \in C$, we can establish $T_{ij}(u)\mathcal{W} \subseteq \mathcal{W}$.

When $M+1 \leq i,j \leq M+N$, we compute $\big[T_{ij}(u), T^{(1)}_{\widehat{m}b}\big] = 0$ for $1 \leq b \leq m$ and $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{jk}T_{il}(u) + \delta_{\bar{\imath}k}\theta_i T_{\bar{l}j}(u)$ for $(k,l) \in C$; thus, by part (i) of Lemma 3.2.12 and the above, we establish $T_{ij}(u)\mathcal{W} \subseteq \mathcal{W}$ for such indices $M+1 \leq i,j \leq M+N$.

Just as in Step 1, the rest of the proof will proceed by systematically showing the inclusion $T_{ij}(u)\mathcal{W} \subseteq \mathcal{W}$ for the remaining indices $(i,j) \in \mathcal{B}_{M|N}$ with similar argument, so we shall only outline how to yield the remaining desired indices.

For indices $\widehat{m}+1 \leq i,j \leq M$, we find $\big[T_{ij}(u), T^{(1)}_{\widehat{m}b}\big] = -\delta_{\bar{j}b}T_{i\widehat{m}}(u)$ for $1 \leq b \leq m$ and $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{\bar{j}l}\theta_k T_{i\bar{k}}(u)$ for $(k,l) \in C$; thus, when $M+1 \leq i \leq M+N$ and $\widehat{m}+1 \leq j \leq M$, we compute $\big[T_{ij}(u), T^{(1)}_{\widehat{m}b}\big] = -\delta_{\bar{j}b}T_{i\widehat{m}}(u)$ for $1 \leq b \leq m$ and $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{\bar{i}k}\theta_i T_{\bar{l}j}(u) - \delta_{\bar{j}l}\theta_k T_{i\bar{k}}(u)$ for $(k,l) \in C$, which shows $T_{ij}(u)\mathcal{W} \subseteq \mathcal{W}$.

For indices $1 \leq i,j \leq m$, we find $\big[T_{ij}(u), T^{(1)}_{\widehat{m}b}\big] = -\delta_{ib}T_{\widehat{m}j}(u)$ for $1 \leq b \leq m$ and $\big[T_{ij}(u), T^{(1)}_{kl}\big] = -\delta_{il}T_{kj}(u)$ for $(k,l) \in C$, which proves $T_{ij}(u)\mathcal{W} \subseteq \mathcal{W}$.

We compute $\big[T_{\widehat{m}\widehat{m}}(u), T^{(1)}_{\widehat{m}b}\big] = T_{\widehat{m}b}(u) + T_{\bar{b}\widehat{m}}(u)$ for $1 \leq b \leq m$ and $\big[T_{\widehat{m}\widehat{m}}(u), T^{(1)}_{kl}\big] = 0$ for $(k,l) \in C$; hence, when $1 \leq i \leq m$, we find $\big[T_{i\widehat{m}}(u), T^{(1)}_{\widehat{m}b}\big] = T_{ib}(u) - \delta_{ib}T_{\widehat{m}\widehat{m}}(u)$ for $1 \leq b \leq m$ and $\big[T_{i\widehat{m}}(u), T^{(1)}_{kl}\big] = 0$ for $(k,l) \in C$, showing $T_{i\widehat{m}}(u)\mathcal{W} \subseteq \mathcal{W}$

For indices $M+1 \leq i \leq M+N$, we find $\big[T_{\widehat{m}i}(u), T^{(1)}_{\widehat{m}b}\big] = T_{\bar{b}i}(u)$ for $1 \leq b \leq m$ and $\big[T_{\widehat{m}i}(u), T^{(1)}_{kl}\big] = -\delta_{ik}T_{\widehat{m}l}(u)$ for $(k,l) \in C$; hence, when $\widehat{m}+1 \leq i \leq M$, we compute $\big[T_{\widehat{m}i}(u), T^{(1)}_{\widehat{m}b}\big] = T_{\bar{b}i}(u) - \delta_{\bar{i}b}T_{\widehat{m}\widehat{m}}(u)$ for $1 \leq b \leq m$ and $\big[T_{\widehat{m}i}(u), T^{(1)}_{kl}\big] = -\delta_{\bar{i}l}\theta_k T_{\widehat{m}\bar{k}}(u)$ for $(k,l) \in C$, showing $T_{\widehat{m}i}(u)\mathcal{W} \subseteq \mathcal{W}$.

For indices $1 \leq i \leq m$ and $M+1 \leq j \leq M+N$, one computes $\big[T_{ij}(u), T^{(1)}_{kl}\big]$ as $\delta_{il}T_{kj}(u) - \delta_{jk}T_{il}(u)$; hence, when we have $1 \leq i \leq m$, $\widehat{m}+1 \leq j \leq M$, then $\big[T_{ij}(u), T^{(1)}_{\widehat{m}b}\big] = -\delta_{ib}T_{\widehat{m}j}(u) - \delta_{\bar{j}b}T_{i\widehat{m}}(u)$ for $1 \leq b \leq m$ and $\big[T_{ij}(u), T^{(1)}_{kl}\big]$ is equal to $-\delta_{il}T_{kj}(u) - \delta_{\bar{j}l}\theta_k T_{i\bar{k}}(u)$ for $(k,l) \in C$, which therefore shows $T_{ij}(u)\mathcal{W} \subseteq \mathcal{W}$ for the remaining indices. In particular, we establish $L(\lambda(u);\alpha,0) = \mathcal{W}$.

We recall the action of $\mathfrak{osp}_{M|N}$ on $\xi$ is determined by $F_{ij}\xi = (-1)^{[i]}T^{(1)}_{ij}\xi$ for indices $1 \leq i,j \leq M+N$. Similar to Step 1, we have $\mathfrak{U}(\mathfrak{osp}_{M|N})\xi = L(\lambda(u);\alpha,0)$, so the quotient of $L(\lambda(u);\alpha,0)$ by its maximal $\mathfrak{osp}_{M|N}$-submodule will be isomorphic to the irreducible representation $V(\lambda^{(1)})$ of $\mathfrak{osp}_{M|N}$ with highest weight $\lambda^{(1)} = (\alpha,\ldots,\alpha,0,\ldots,0)$ where the first $m$ many terms are $\alpha$. If $L(\lambda(u);\alpha,0)$ is finite-dimensional, then $V(\lambda^{(1)})$ is so which forces the requirement $\alpha \in \frac{1}{2}\mathbb{Z}^+$.

Conversely, suppose $\alpha \in \frac{1}{2}\mathbb{Z}^+$ is such that the highest weight $\mu = (\alpha,\ldots,\alpha,0,\ldots,0)$ of the above form makes the irreducible $\mathfrak{osp}_{M|N}$-module $V(\mu)$ be finite-dimensional. If $\zeta$ denotes the highest weight vector of $V(\mu)$, we can compute

$$V(\mu) = \operatorname{span}_{\mathbb{C}}\big\{\big(F_{\widehat{m}b_1}\cdots F_{\widehat{m}b_s}\big)\big(F_{k_1l_1}\cdots F_{k_rl_r}\big)\zeta \mid 1 \leq b_j \leq m,\ (k_i,l_i) \in C,\ s,r \in \mathbb{Z}^+\big\},$$

where $C$ and '$\preceq$' is defined similar to before such that $F_{\widehat{m}b_j} \preceq F_{\widehat{m}b_{j+1}}$, $F_{k_il_i} \preceq F_{k_{i+1}l_{i+1}}$

for $1 \leq j \leq s-1$, $1 \leq i \leq r-1$. In particular, we deduce there is an isomorphism $L(\lambda(u); \alpha, 0) \cong V(\mu)$ of $\mathfrak{osp}_{M|N}$-modules. □

**Corollary 3.2.14.** *Suppose $M, N \geq 2$ and let $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq \beta$. When $M = 2$, then $\dim L(\lambda(u); \alpha, \beta) \leq 2^N$. Otherwise when $M \geq 3$, then $L(\lambda(u); \alpha, \beta)$ is finite-dimensional if and only if the irreducible $\mathfrak{osp}_{M|N}$-module $V(\mu)$ is finite-dimensional, where*

$$\mu = (\underbrace{\alpha - \beta, \ldots, \alpha - \beta}_{m}, \underbrace{0, \ldots, 0}_{n}).$$

*Necessarily, $\alpha - \beta \in \frac{1}{2}\mathbb{Z}^+$. When $M$ is even in this case, then $\dim L(\lambda(u); \alpha, \beta) \leq 2^{mN}$.*

### 3.2.4 Type II fundamental representations

In this subsection, we show that many type II fundamental representations $L(\lambda(u); i \colon \gamma)$ as in Definition 3.2.9 are finite-dimensional. As noted in the previous subsection, the pullback of $L(\lambda(u); i \colon \gamma)$ by the shift automorphism $\tau_a$ (2.2.10) will yield a module $\tau_a^*(L(\lambda(u); i \colon \gamma))$ isomorphic to $L(\lambda(u-a); i \colon \gamma - a)$. In particular, the dimensions of $L(\lambda(u); i \colon \gamma)$ and $L(\lambda(u-a); i \colon \gamma - a)$ coincide, which means it suffices to prove that $\dim L(\lambda(u); i \colon \gamma) < \infty$ for any $\gamma \in \mathbb{C}$ with highest weight $\lambda(u)$ satisfying the consistency conditions in Proposition 3.2.1.

We construct two families of representations of $\mathrm{X}(\mathfrak{osp}_{M|N})$ by tensoring vector representations of the form (2.3.9) in suitable ways. Before doing so, we construct vector representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ that produce highest weight representations which will be utilized later in the section. As a direct analogue of (2.3.7), there is an $R$-matrix representation of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ given by the assignment

$$\dot{R} \colon \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \operatorname{End} \mathbb{C}^{m|n}, \quad t(u) \mapsto \dot{R}(u).$$

Using the analogues of the superalgebra anti-automorphisms (2.2.12) and (2.2.14) for $\mathrm{Y}(\mathfrak{gl}_{m|n})$, so one can yield a representation

$$\dot{\varrho} \colon \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \operatorname{End} \mathbb{C}^{m|n}, \quad t(u) \mapsto \dot{R}^{st_1}(-u).$$

On the level of power series, such representation takes the form

$$\dot{\varrho}\colon t_{ij}(u) \mapsto \delta_{ij}\,\mathrm{id} + \frac{(-1)^{[i]}E_{ij}}{u}$$

and we call $\dot{\varrho}$ the *vector representation of* $\mathrm{Y}(\mathfrak{gl}_{m|n})$. Finally, postcomposing $\dot{\varrho}$ with an analogue of the automorphism $\tau_a$ as in (2.2.10) will result in a representation of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ given by

$$\dot{\varrho}_a\colon\ \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{End}\,\mathbb{C}^{m|n}, \quad t(u) \mapsto \dot{R}^{st_1}(a-u) \tag{3.2.23}$$

for any $a \in \mathbb{C}$. On the level of power series, such representation takes the form

$$\dot{\varrho}_a\colon t_{ij}(u) \mapsto \delta_{ij}\,\mathrm{id} + \frac{(-1)^{[i]}E_{ij}}{u-a}.$$

We call $\dot{\varrho}_a$ the *vector representation of* $\mathrm{Y}(\mathfrak{gl}_{m|n})$ *at* $a \in \mathbb{C}$. For $d \in \mathbb{Z}^+$, tensoring these vector representations from levels $d-1$ to $0$ gives rise to a representation of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ on $(\mathbb{C}^{M|N})^{\otimes d}$ called the *vector representation of* $\mathrm{Y}(\mathfrak{gl}_{m|n})$ *from levels* $d-1$ *to* $0$, denoted $\dot{\varrho}_{(d-1)\to 0} := (\bigotimes_{i=1}^{d} \dot{\varrho}_{d-i}) \circ \Delta_{d-1}$ where

$$\begin{aligned} \dot{\varrho}_{(d-1)\to 0}\colon\ \mathrm{Y}(\mathfrak{gl}_{m|n}) &\to \mathrm{End}(\mathbb{C}^{m|n})^{\otimes d} \\ t(u) &\mapsto \textstyle\prod_{i=1}^{d} \dot{R}_{1,i+1}^{st_1}((d-i)-u). \end{aligned} \tag{3.2.24}$$

We shall show that there exists a certain invariant subspace of $(\mathbb{C}^{m|n})^{\otimes d}$ that is a highest weight representation for the super Yangian.

For any super vector space $V$, we let $\epsilon\colon \mathfrak{S}_d \times V^{\otimes d} \to \mathbb{Z}_2$ denote the map defined by $\epsilon(\sigma, v) = \sum_{(i,j)\in \mathrm{Inv}(\sigma)} [v_{\sigma(i)}][v_{\sigma(j)}]$ where $v = v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$ is homogeneous decomposable tensor and $\mathrm{Inv}(\sigma) = \{(i,j) \mid i<j,\ \sigma(i) > \sigma(j)\}$ is the set of inversions. The Koszul sign is defined as the value $(-1)^{\epsilon(\sigma,v)}$. Accordingly, there is a representation of the symmetric group $\mathfrak{S}_d$ on $V^{\otimes d}$ via the formula

$$\sigma^{-1} \cdot (v_1 \otimes \cdots \otimes v_d) = (-1)^{\epsilon(\sigma,v)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}. \tag{3.2.25}$$

When $V = \mathbb{C}^{m|n}$, such representation maps a transposition $(a\ b) \in \mathfrak{S}_d$ with $a<b$ to

the operator

$$P_{ab} = \sum_{i,j=1}^{m+n} (-1)^{[j]} \operatorname{id}^{\otimes(a-1)} \otimes E_{ij} \otimes \operatorname{id}^{\otimes(b-a-1)} \otimes E_{ji} \otimes \operatorname{id}^{\otimes(d-b)} .$$

This representation lifts to a corresponding representation of the group algebra $\mathbb{C}\mathfrak{S}_d$ on the same tensor space. Considering the anti-symmetrizer $a^{(d)} \in \mathbb{C}\mathfrak{S}_d$ given by the formula $a^{(d)} = \frac{1}{d!}\sum_{\sigma\in\mathfrak{S}}(\operatorname{sgn}\sigma)\sigma$, we may consider the image of its action on such tensor space, denoted $A^{(d)}(\mathbb{C}^{m|n})^{\otimes d}$, where $A^{(d)}$ is the image of $a^{(d)}$ under the representation $\mathbb{C}\mathfrak{S}_d \to \operatorname{End}(\mathbb{C}^{m|n})^{\otimes d}$.

**Proposition 3.2.15** (A. Molev)**.** *Let $m, n \geq 1$. The subspace $A^{(d)}(\mathbb{C}^{m|n})^{\otimes d} \subset (\mathbb{C}^{m|n})^{\otimes d}$ is invariant under the vector representation* (3.2.24) *of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ from levels $d-1$ to 0. Furthermore,*

(i) *If $d \leq m$, setting $x_d := e_1 \otimes \cdots \otimes e_d$ and $\zeta_d := d!A^{(d)}(x_d)$ provides the relations*

$$t_{ij}(u)\,\zeta_d = 0 \quad \text{for} \quad \begin{cases} i \in \{1,2,\dots,d\},\ j \in \{1,2,\dots,m+n\} \setminus \{i\}, \\ i \in \{d+1,\dots,m+n\},\ j \in \{d+1,\dots,m+n\} \setminus \{i\}, \end{cases}$$

*and*

$$t_{ii}(u)\,\zeta_d = \begin{cases} \dfrac{u+1}{u}\zeta_d & \text{for} \quad 1 \leq i \leq d, \\ \zeta_d & \text{for} \quad d+1 \leq i \leq m+n. \end{cases}$$

(ii) *If $d = m+1$ and $m+1 \leq k \leq m+n$, setting $x_{m,k} := e_1 \otimes \cdots \otimes e_m \otimes e_k$ and $\zeta_{m,k} := (m+1)!A^{(m+1)}(x_{m,k})$ provides the relations*

$$t_{ij}(u)\,\zeta_{m,k} = 0 \quad \text{for} \quad \begin{cases} i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,m+n\} \setminus \{i\}, \\ i = k;\ j \in \{1,2,\dots,m\}, \\ i \in \{m+1,\dots,m+n\},\ j \in \{m+1,\dots,m+n\} \setminus \{i,k\}, \end{cases}$$

*and*

$$t_{ii}(u)\,\zeta_{m,k} = \begin{cases} \dfrac{u+1}{u}\zeta_{m,k} & \text{for} \quad i \in \{1,2,\dots,m\}, \\ \zeta_{m,k} & \text{for} \quad i \in \{m+1,\dots,m+n\} \setminus \{k\}, \\ \dfrac{u-1}{u}\zeta_{m,k} & \text{for} \quad i = k. \end{cases}$$

*Proof.* We refer the reader to Appendix A in [Mol22b] for the original proof of these results and we shall reproduce an outline of the proof here. By virtue of the fusion procedure, c.f. [Mol07, §6.4], there is an equality in the space $\mathrm{End}(\mathbb{C}^{m|n})^{\otimes(d+1)}$ given by

$$\left(\prod_{i=1}^{d} \dot{R}_{1,i+1}(u+i-1)\right)(\mathrm{id}\otimes A^{(d)}) = (\mathrm{id}\otimes A^{(d)})\left(\mathrm{id}^{\otimes(d+1)} - \frac{\sum_{i=1}^{d} P_{1,i+1}}{u}\right).$$

Applying $(-)^{st}$ to the first tensor factor of the equation above and substituting $u \mapsto -u$ will yield the equation

$$\left(\prod_{i=1}^{d} \dot{R}_{1,d+2-i}((d-i)-u)\right)(\mathrm{id}\otimes A^{(d)}) = (\mathrm{id}\otimes A^{(d)})\left(\mathrm{id}^{\otimes(d+1)} + \frac{\sum_{i=1}^{d} P_{1,i+1}^{st_1}}{u}\right).$$

Considering that the permutation $\omega \in \mathfrak{S}_d$ mapping $a \mapsto d+1-a$ for integers $1 \le a \le d$ can be written as $\omega = (1\ d)(2\ d-1)\cdots(\lceil \frac{d}{2}\rceil\ \lfloor \frac{d}{2}\rfloor+1)$, we can describe its image $P_\omega$ in $\mathrm{End}(\mathbb{C}^{m|n})^{\otimes d}$. Multiplying the above equation on the left by $\mathrm{id}\otimes P_\omega$ therefore yields

$$\left(\prod_{i=1}^{d} \dot{R}_{1,i+1}((d-i)-u)\right)(\mathrm{id}\otimes A^{(d)}) = (\mathrm{sgn}\,\omega)(\mathrm{id}\otimes A^{(d)})\left(\mathrm{id}^{\otimes(d+1)} + \frac{\sum_{i=1}^{d} P_{1,i+1}^{st_1}}{u}\right)$$

since $\omega a^{(d)} = (\mathrm{sgn}\,\omega)\, a^{(d)}$.

To prove (i), the above argument shows that

$$t_{ij}(u)\zeta_d = d!A^{(d)}\left(\delta_{ij}x_d + \frac{(-1)^{[i]}\sum_{a=1}^{d}\left(\mathrm{id}^{\otimes(a-1)}\otimes E_{ij}\otimes \mathrm{id}^{\otimes(d-a)}\right)(x_d)}{u}\right).$$

Since any decomposable tensor that contains an identical vector in two separate tensor factors lies in the kernel of $A^{(d)}$, we can conclude $t_{ij}(u)\zeta_d = 0$ for $i \in \{1,2,\dots,d\}$, $j \in \{1,2,\dots,m+n\}\setminus\{i\}$ and $i \in \{d+1,\dots,m+n\}$, $j \in \{d+1,\dots,m+n\}\setminus\{i\}$. Moreover, one can immediately verify $t_{ii}(u)\zeta_d = (1+u^{-1})\zeta_d$ for $1 \le i \le d$ and $t_{ii}(u)\zeta_d = \zeta_d$ for $d+1 \le i \le m+n$.

For (ii), the action $t_{ij}(u)\zeta_{m,k}$ is similarly given by

$$(m+1)!A^{(m+1)}\left(\delta_{ij}x_{m,k} + \frac{(-1)^{[i]}\sum_{a=1}^{m+1}\left(\mathrm{id}^{\otimes(a-1)}\otimes E_{ij}\otimes \mathrm{id}^{\otimes(m+1-a)}\right)(x_{m,k})}{u}\right).$$

With similar reasoning to before, we can conclude that $t_{ij}(u)\zeta_{m,k} = 0$ for indices satisfying either of the conditions $i \in \{1,2,\dots,m\}$, $j \in \{1,2,\dots,m{+}n\}\setminus\{i\}$, or $i \in \{k\}$, $j \in \{1,2,\dots,m\}$, or $i \in \{m{+}1,\dots,m{+}n\}$, $j \in \{m{+}1,\dots,m{+}n\} \setminus \{i,k\}$. Further, we can conclude $t_{ii}(u)\zeta_{m,k} = (1+u^{-1})\zeta_{m,k}$ for $i \in \{1,2,\dots,m\}$, $t_{ii}(u)\zeta_{m,k} = \zeta_{m,k}$ for $i \in \{m{+}1,\dots,m{+}n\} \setminus \{k\}$, and $t_{kk}(u)\zeta_{m,k} = (1-u^{-1})\zeta_{m,k}$. □

**Corollary 3.2.16** (A. Molev)**.** *For $d \le m$, the submodule $\mathrm{Y}(\mathfrak{gl}_{m|n})\zeta_d \subseteq A^{(d)}(\mathbb{C}^{m|n})^{\otimes d}$ is highest weight with highest weight vector $\zeta_d$ and highest weight $\lambda(u) = (\lambda_i(u))_{i=1}^{m+n}$, where $\lambda_i(u) = 1+u^{-1}$ for $1 \le i \le d$ and $\lambda_i(u) = 1$ for $d{+}1 \le i \le m{+}n$.*

*Similarly, $\mathrm{Y}(\mathfrak{gl}_{m|n})\zeta_{m,m+1} \subseteq A^{(m+1)}(\mathbb{C}^{m|n})^{\otimes(m+1)}$ is a highest weight module with highest weight vector $\zeta_{m,m+1}$ and highest weight $\lambda(u) = (\lambda_i(u))_{i=1}^{m+n}$, where $\lambda_i(u) = 1{+}u^{-1}$ for $1 \le i \le m$, $\lambda_{m+1}(u) = 1-u^{-1}$, and $\lambda_i(u) = 1$ for $m{+}2 \le i \le m{+}n$.*

We shall now assume $m = \lfloor \frac{M}{2} \rfloor$, $\widehat{m} = \lceil \frac{M}{2} \rceil$, and $n = \frac{N}{2}$ for the remainder of the section. We recall the vector representation (2.3.9) of $\mathrm{X}(\mathfrak{osp}_{M|N})$ on $\mathbb{C}^{M|N}$ at level $a \in \mathbb{C}$ defined by $\varrho_a \colon T(u) \mapsto R^{st}(a-u)$. On the level of power series, such representation is given by

$$\varrho_a \colon T_{ij}(u) \mapsto \delta_{ij}\,\mathrm{id} + \frac{(-1)^{[i]}E_{ij}}{u-a} - \frac{(-1)^{[i][j]}\theta_i\theta_j E_{\bar{j}\bar{\imath}}}{u+\kappa-a}, \quad \text{where} \quad \varrho = \varrho_0,$$

and we shall let the juxtaposition $T_{ij}(u)v$ for $v \in \mathbb{C}^{M|N}$ denote the action $\varrho(T_{ij}(u))v$. A notable property for the vector representation is that

$$\varrho(T_{ij}(-u-\kappa-c)) = (-1)^{[i][j]+[j]}\theta_i\theta_j\,\varrho(T_{\bar{j}\bar{\imath}}(u+c)) \quad \text{for any} \quad c \in \mathbb{C}.$$

Tensoring these vector representations from levels 0 to $1-d$ gives rise to a representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ on $(\mathbb{C}^{M|N})^{\otimes d}$ called the *vector representation of* $\mathrm{X}(\mathfrak{osp}_{M|N})$ *from levels* 0 *to* $1-d$, denoted $\varrho_{0\to(1-d)} := (\bigotimes_{i=0}^{d-1} \varrho_{-i}) \circ \Delta_{d-1}$, and written

$$\begin{aligned} \varrho_{0\to(1-d)} \colon\ \mathrm{X}(\mathfrak{osp}_{M|N}) &\to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes d} \\ T(u) &\mapsto \textstyle\prod_{i=1}^{d} R^{st_1}_{1,i+1}((1-i)-u). \end{aligned} \tag{3.2.26}$$

Letting $\{e_i\}_{i=1}^{M+N}$ denote the standard basis for $\mathbb{C}^{M|N}$ with $\mathbb{Z}_2$-grade $[e_i] = [i]$, any integer

$1 \le d \le m$ gives rise to the element

$$\xi_d := \sum_{\sigma \in \mathfrak{S}_d} (\operatorname{sgn} \sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(d)} \in (\mathbb{C}^{M|N})^{\otimes d}$$

and we claim that $\xi_d$ generates a highest weight module over $\mathrm{X}(\mathfrak{osp}_{M|N})$.

**Proposition 3.2.17.** *Suppose $M \ge 2$. Given $1 \le d \le m$, $\xi_d \in (\mathbb{C}^{M|N})^{\otimes d}$, and the module action (3.2.26) on $(\mathbb{C}^{M|N})^{\otimes d}$, the $\mathrm{X}(\mathfrak{osp}_{M|N})$-submodule generated by $\xi_d$ is a highest weight module with highest weight vector $\xi_d$ and highest weight $(\lambda_k(u))_{k=1}^{M+N}$ given by*

$$\lambda_k(u) = \begin{cases} \dfrac{u+d}{u+d-1} & \text{if } \ 1 \le k \le d, \\ \dfrac{u+\kappa-1}{u+\kappa} & \text{if } \ M-d+1 \le k \le M, \\ 1 & \text{otherwise.} \end{cases} \tag{3.2.27}$$

*The quotient of $\mathrm{X}(\mathfrak{osp}_{M|N})\xi_d$ by its maximal submodule will be isomorphic to $L(\lambda(u))$ with $\dim L(\lambda(u)) < \infty$. When $d < m$, then $L(\lambda(u))$ will be the type II fundamental representation $L(\lambda(u); d : d-1)$. Otherwise when $d = m$ and $N \ge 2$, then $L(\lambda(u))$ will be the type I fundamental representation $L(\lambda(u); m, m-1)$.*

*Proof.* Let us consider how $T_{ij}(u)$ acts on tensor products $e_{p_1} \otimes \cdots \otimes e_{p_d}$ of even basis vectors $e_{p_1}, \dots, e_{p_d}$ of $\mathbb{C}^{M|N}$ with indices satisfying $1 \le p_1, \dots, p_d \le m$. Since $\varrho_{-a}(T_{ij}(u)) = \varrho(T_{ij}(u+a))$ for any $a \in \mathbb{C}$, such action is described by the formula

$$\begin{aligned} &T_{ij}(u)(e_{p_1} \otimes \cdots \otimes e_{p_d}) \\ &\quad = \sum_{a_1,\dots,a_{d-1}=1}^{M+N} T_{ia_1}(u) e_{p_1} \otimes T_{a_1a_2}(u+1) e_{p_2} \otimes \cdots \otimes T_{a_{d-1}j}(u+(d-1)) e_{p_d}. \end{aligned} \tag{3.2.28}$$

With suitable restrictions on the index $i$, many of the terms in the above sum will be zero. In particular, the summation indices $a_1, \dots, a_{d-1}$ in the formula (3.2.28) can be restricted to $1 \le a_1, \dots, a_{d-1} \le m$ when $1 \le i \le \widehat{m}$ or restricted to $1 \le a_1, \dots, a_{d-1} \le m$ and $M+1 \le a_1, \dots, a_{d-1} \le M+N$ when $M+1 \le i \le M+N$.

In particular, we can conclude $T_{ij}(u)(e_{p_1} \otimes \cdots \otimes e_{p_d}) = 0$ for indices $(i,j)$ lying in $\Gamma_{0,1} \cup \Gamma_{1,0} \cup \Gamma_{1,1}$ and those $(i,j) \in \Gamma_{0,0}$ such that $1 \le i \le \widehat{m}$ and $m+1 \le j \le M$.

Moreover, for indices $M+1 \leq i = j \leq M+N$, and $i = j = \widehat{m}$ when $M$ is odd, we have $T_{ii}(u)(e_{p_1} \otimes \cdots \otimes e_{p_d}) = e_{p_1} \otimes \cdots \otimes e_{p_d}$, so (3.2.27) is verified for these specified indices.

It remains to determine the action $T_{ij}(u)\xi_d$ for indices satisfying $1 \leq i \leq j \leq m$ and $\widehat{m}+1 \leq i \leq j \leq M$. Similar to before, we observe that restricting $\widehat{m}+1 \leq j \leq M$ infers that the summation indices $a_1, \ldots, a_{d-1}$ in the formula (3.2.28) can be limited to $\widehat{m}+1 \leq a_1, \ldots, a_{d-1} \leq M$. Allowing $\omega \in \mathfrak{S}_d$ to denote the involutive permutation $\omega \colon a \mapsto d+1-a$ for integers $1 \leq a \leq d$, we set $P_\omega$ to denote its image under the representation $\mathbb{C}\mathfrak{S}_d \to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes d}$ described by the action (3.2.25).

When $1 \leq i, j \leq m$, the restrictions on the indices $a_1, \ldots, a_{d-1}$ along with the property $\varrho(T_{ij}(-u-\kappa-c)) = \varrho(T_{\bar{j}\bar{i}}(u+c))$ infers that the conjugation of the action of $T_{ij}(-u-\kappa-(d-1))$ with $P_\omega$ on tensor products of even basis vectors $e_{p_1} \otimes \cdots \otimes e_{p_d}$ of $\mathbb{C}^{M|N}$ with $1 \leq p_1, \ldots, p_d \leq m$ can be described by the formula

$$\big(P_\omega \cdot T_{ij}(-u-\kappa-(d-1)) \cdot P_\omega\big)(e_{p_1} \otimes \cdots \otimes e_{p_d}) = T_{\bar{j}\bar{i}}(u)(e_{p_1} \otimes \cdots \otimes e_{p_d}).$$

The above relation shows that the action of $T_{ij}(-u-\kappa-(d-1))$ on $\xi_d$ for indices $1 \leq i, j \leq m$ determines the action of $T_{\bar{j}\bar{i}}(u)$ on $\xi$. In particular, when $1 \leq i, j \leq m$, the action (3.2.28) takes the same form as its $\mathrm{Y}(\mathfrak{gl}_m)$-counterpart as in [AMR06, §5.3]; hence, $T_{ij}(u)\xi_d = 0$ is true for $1 \leq i < j \leq m$, so $T_{ij}(u)\xi_d = 0$ for all $(i, j) \in \Lambda^+$.

Moreover, the $\mathrm{Y}(\mathfrak{gl}_M)$ case implies the formula (3.2.27) for values $1 \leq k \leq m$. To yield the remaining relations for $m+1 \leq k \leq M$, we use $P_\omega(\xi_d) = (\mathrm{sgn}\,\omega)\xi_d$ to get

$$T_{kk}(u)\xi_d = \big(P_\omega \cdot T_{\bar{k}\bar{k}}(-u-\kappa-(d-1)) \cdot P_\omega\big)\xi_d = \frac{u+\kappa-1}{u+\kappa}\xi_d.$$

Thus, the irreducible quotient $\mathrm{X}(\mathfrak{osp}_{M|N})\xi_d/\mathcal{M}$ by its maximal submodule $\mathcal{M}$ will be a finite-dimensional highest weight representation with highest weight vector $\xi_d \bmod \mathcal{M}$ and highest weight $\lambda(u) = \big(\lambda_k(u)\big)_{k=1}^{M+N}$, where $\lambda_k(u)\xi_d = T_{kk}(u)\xi_d$ as in (3.2.27). The Drinfel'd polynomial relations can be verified directly. □

We shall now construct another family of representations of $\mathrm{X}(\mathfrak{osp}_{M|N})$; however, these constructions will not give rise to type II fundamental representations unless $M = 2$. Nonetheless, we provide the general framework in hope that the following constructions can be used or modified to generate the the remaining type II fundamental representations when $M \geq 3$ in future research.

We assume $M \geq 2$ for the remainder of the subsection. For $M+1 \leq k \leq M+N$, along with $k = \widehat{m}$ when $M$ is odd, we define the element

$$w_k := \sum_{\sigma \in \mathfrak{S}^k_{m+1}} (\operatorname{sgn} \sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(m)} \otimes e_{\sigma(k)} \in (\mathbb{C}^{M|N})^{\otimes(m+1)}, \tag{3.2.29}$$

where $\mathfrak{S}^k_{m+1}$ is the symmetric group on the symbols $\{1, \ldots, m\} \cup \{k\}$. Recalling the vector representation of $\mathrm{X}(\mathfrak{osp}_{M|N})$ at level $a \in \mathbb{C}$ given by $\varrho_a \colon T(u) \mapsto R^{st_1}(a-u)$, see (2.3.9), we can tensor these representations from levels $m$ to 0, which we denote $\varrho_{m\to 0} := (\bigotimes_{i=0}^m \varrho_{m-i}) \circ \Delta_{m-1}$, to yield the representation

$$\begin{aligned} \varrho_{m\to 0} \colon \mathrm{X}(\mathfrak{osp}_{M|N}) &\to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes(m+1)} \\ T(u) &\mapsto \textstyle\prod_{i=1}^d R^{st_1}_{1,i+1}((m-i)-u). \end{aligned} \tag{3.2.30}$$

We now consider how the above representation acts on the elements (3.2.29).

**Lemma 3.2.18.** *Given integers $M+1 \leq k \leq M+N$, $w_k \in (\mathbb{C}^{M|N})^{\otimes(m+1)}$, and the module action (3.2.30), we have the following relations:*

(i) $T_{ij}(u) w_k = 0$ *for any indices $1 \leq i, j \leq M+N$ satisfying*

$$\begin{gathered} i \in \{1, 2, \ldots, m\},\ j \in \{1, 2, \ldots, M+N\} \setminus \{i\}; \\ i \in \{\widehat{m}\},\ j \in \{\widehat{m}+1, \ldots, M+N\} \setminus \{k\}; \qquad i \in \{k\},\ j \in \{1, 2, \ldots, m\}; \\ i \in \{\widehat{m}+1, \ldots, M+N\},\ j \in \{\widehat{m}+1, \ldots, M\} \setminus \{i\}; \\ i \in \{M+1, \ldots, M+N\} \setminus \{\bar{k}\},\ j \in \{m+1, \ldots, M\}; \\ i \in \{M+1, \ldots, M+N\} \setminus \{\bar{k}\},\ j \in \{M+1, \ldots, M+N\} \setminus \{i, k\}. \end{gathered}$$

(ii) $T_{ii}(u) w_k$ *is an element in $\mathbb{C}[\![u^{-1}]\!] w_k$ as described by*

$$T_{ii}(u) w_k = \begin{cases} \dfrac{u+1}{u} w_k & \textit{if} \quad i \in \{1, 2, \ldots, m\}, \\ \dfrac{u+\kappa-m-1}{u+\kappa-m} w_k & \textit{if} \quad i \in \{\widehat{m}+1, \ldots, M\}, \\ \dfrac{u-1}{u} w_k & \textit{if} \quad i = k, \\ w_k & \textit{otherwise.} \end{cases}$$

*Proof.* We will consider how $T_{ij}(u)$ acts on decomposable tensors whose components are comprised of $e_k$ and $m$ even basis vectors $e_{p_1}, \dots, e_{p_m}$ of $\mathbb{C}^{M|N}$ with indices satisfying $1 \le p_1, \dots, p_m \le m$. Since $\varrho_a(T_{ij}(u)) = \varrho(T_{ij}(u-a))$ for $a \in \mathbb{C}$, such action is described by the formula

$$\begin{aligned} &T_{ij}(u)(e_{p_1} \otimes \cdots e_{p_{s-1}} \otimes e_k \otimes e_{p_s} \otimes \cdots \otimes e_{p_m}) \\ &\quad = \sum_{a_1,\dots,a_m=1}^{M+N} (-1)^{(1+\delta_{s-1,m})([a_s]+[j])[k]} T_{ia_1}(u-m)e_{p_1} \otimes \cdots \\ &\qquad\qquad \cdots \otimes T_{a_{s-1}a_s}(u-(m+1-s))e_k \otimes \cdots \otimes T_{a_m j}(u)e_{p_m}. \end{aligned} \tag{3.2.31}$$

With suitable restriction on the index $i$, many of the terms in the above sum will be zero. In particular, when $1 \le i \le \widehat{m}$ or $M+1 \le i \le M+N$ such that $\bar{\imath} \ne k$, the summation indices $a_1, \dots, a_m$ in the formula (3.2.31) can be restricted to $1 \le a_1, \dots, a_m \le m$ and $M+1 \le a_1, \dots, a_m \le M+N$. Thus, we have

$$T_{ii}(u)w_k = w_k \quad \text{for} \quad \begin{cases} i = \widehat{m} \ \text{ when } M \text{ is odd}, \\ i \in \{M+1, \dots, M+N\} \setminus \{k, \bar{k}\}. \end{cases}$$

If one further assumes $i \ne j$, then the summation (3.2.31) can be written as

$$\begin{aligned} &\sum_{r=1}^{s-1} \sum_{a_r=1}^{m} \sum_{t=s}^{m} \Big( \sum_{a_t=1}^{m} + \sum_{\substack{a_t=\cdots=a_s=k,\\ 1\le a_{t+1},\dots,a_m \le m}} \Big) (-1)^{(1+\delta_{s-1,m})([a_s]+[j])[k]} T_{ia_1}(u-m)e_{p_1} \otimes \cdots \\ &\qquad\qquad \cdots \otimes T_{a_{s-1}a_s}(u-(m+1-s))e_k \otimes \cdots \otimes T_{a_m j}(u)e_{p_m}, \end{aligned}$$

so we can deduce

$$T_{ij}(u)w_k = 0 \quad \text{for} \quad \begin{cases} i \in \{1, 2, \dots, \widehat{m}\},\ j \in \{\widehat{m}+1, \dots, M+N\} \setminus \{k\}, \\ i \in \{M+1, \dots, M+N\} \setminus \{\bar{k}\},\ j \in \{m+1, \dots, M+N\} \setminus \{i, k\}. \end{cases}$$

When $1 \le i \le \widehat{m}$ or $M+1 \le i \le M+n$, restraining $M+1 \le k \le M+n$ allows the summation indices $a_1, \dots, a_m$ in the formula (3.2.31) to be restricted further to $1 \le a_1, \dots, a_m \le m$ and $M+1 \le a_1, \dots, a_m \le M+n$. Hence, under the embedding

$$\nu\colon\ \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{X}(\mathfrak{osp}_{M|N}), \quad t_{ij}(u) \mapsto T_{\nu(i)\nu(j)}(u),$$

where $\nu(i) = i$ for $1 \le i \le m$ and $\nu(i) = \widehat{m}+i$ for $m+1 \le i \le m+n$, the map

$e_i \mapsto e_{\nu(i)}$ induces a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module isomorphism $\mathrm{Y}(\mathfrak{gl}_{m|n})\zeta_{m,k} \cong \mathrm{Y}(\mathfrak{gl}_{m|n})w_k$ for indices $M+1 \leq k \leq M+n$.

Alternatively, when $1 \leq i \leq \widehat{m}$ or $M+n+1 \leq i \leq M+N$, restraining the index $M+n+1 \leq k \leq M+N$ permits the summation indices $a_1, \ldots, a_m$ in the formula (3.2.31) to be restricted further to $1 \leq a_1, \ldots, a_m \leq m$ and $M+n+1 \leq a_1, \ldots, a_m \leq M+N$. Hence, under another embedding

$$\nu'\colon\ \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{X}(\mathfrak{osp}_{M|N}), \quad t_{ij}(u) \mapsto T_{\nu'(i)\nu'(j)}(u),$$

where $\nu'(i) = i$ for $1 \leq i \leq m$ and $\nu'(i) = \widehat{m}+n+i$ for $m+1 \leq i \leq m+n$, the map $e_i \mapsto e_{\nu'(i)}$ induces a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module isomorphism $\mathrm{Y}(\mathfrak{gl}_{m|n})\zeta_{m,k} \cong \mathrm{Y}(\mathfrak{gl}_{m|n})w_k$ for indices $M+n+1 \leq k \leq M+N$. Therefore, via these two embeddings and the calculations performed earlier in the proof, we can conclude

$$T_{ij}(u)w_k = 0 \quad \text{for} \quad \begin{cases} i \in \{1,2,\ldots,m\},\ j \in \{1,2,\ldots,M+N\} \setminus \{i\}, \\ i = \widehat{m};\ j \in \{\widehat{m}+1,\ldots,M+N\} \setminus \{k\}, \\ i = k;\ j \in \{1,2,\ldots,m\}, \\ i \in \{M+1,\ldots,M+N\} \setminus \{\bar{k}\},\ j \in \{m+1,\ldots,M+N\} \setminus \{i,k\}. \end{cases}$$

along with

$$T_{ii}(u)w_k = \begin{cases} \dfrac{u+1}{u} w_k & \text{if} \quad i \in \{1,2,\ldots,m\}, \\ w_k & \text{if} \quad i = \widehat{m} \ \text{ and } \ M \text{ is odd}, \\ w_k & \text{if} \quad i \in \{M+1,\ldots,M+N\} \setminus \{k\}, \\ \dfrac{u-1}{u} w_k & \text{if} \quad i = k. \end{cases}$$

Let us now derive the the remaining relations. Assuming $\widehat{m}+1 \leq j \leq M$ allows the summation indices $a_1, \ldots, a_m$ to be restricted to $\widehat{m}+1 \leq a_1, \ldots, a_m \leq M+N$. In particular, we can write the sum (3.2.31) as

$$\sum_{r=s}^{m} \sum_{a_r=\widehat{m}+1}^{M} \sum_{t=1}^{s-1} \Big( \sum_{a_t=\widehat{m}+1}^{M} + \sum_{\substack{a_t=\cdots=a_{s-1}=\bar{k}, \\ \widehat{m}+1 \leq a_1,\ldots,a_{t-1} \leq M}} \Big) (-1)^{(1+\delta_{s-1,m})([a_s]+[j])[k]} T_{ia_1}(u-m)e_{p_1} \otimes \cdots$$
$$\cdots \otimes T_{a_{s-1}a_s}(u-(m+1-s))e_k \otimes \cdots \otimes T_{a_m j}(u)e_{p_m}.$$

Letting $\omega \in \mathfrak{S}_{m+1}$ denote the involutive permutation $\omega\colon a \mapsto m+2-a$ for integers $1 \leq a \leq m+1$, we set $P_\omega$ to denote its image under the algebra representation $\mathbb{C}\mathfrak{S}_{m+1} \to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes(m+1)}$ described by the action (3.2.25). If $\widehat{m}+1 \leq i \leq M+N$ and $\widehat{m}+1 \leq j \leq M$, the restrictions on the indices $a_1, \ldots, a_{d-1}$ along with the property $\varrho(T_{ij}(-u-\kappa-c)) = (-1)^{[i][j]+[j]}\theta_i\theta_j\varrho(T_{\bar{j}\bar{i}}(u+c))$ infers that the conjugation of the action of $T_{ij}(-u-\kappa-(d-1))$ with $P_\omega$ on $w_k$ can be described by the formula

$$\big(P_\omega \cdot T_{ij}(u) \cdot P_\omega\big)w_k = \theta_i T_{\bar{j}\bar{i}}(-u-\kappa+m)w_k$$

Hence, for such indices $\widehat{m}+1 \leq i \leq M+N$, $\widehat{m}+1 \leq j \leq M$ satisfying $i \neq j$, we use the fact that $P_\omega^2 = \mathrm{id}^{\otimes(m+1)}$ to compute $T_{ij}(u)w_k = 0$, yielding the remaining relations for (i). Furthermore, when $\widehat{m}+1 \leq i \leq M$ one computes

$$T_{ii}(u)w_k = (\operatorname{sgn}\omega)\big(T_{ii}(u)\cdot P_\omega\big)w_k = \frac{-u-\kappa+m+1}{-u-\kappa+m}w_k = \frac{u+\kappa-m-1}{u+\kappa-m}w_k$$

which establishes the remaining relations for (ii). □

Allowing $\dot{T}_{ij}(u)$ to denote a generating series for the extended Yangian $\mathrm{X}(\mathfrak{osp}_{0|N})$ and $\acute{T}_{ij}(u)$ to denote a generating series for the extended Yangian $\mathrm{X}(\mathfrak{osp}_{1|N})$, we have the following proposition:

**Proposition 3.2.19.** *Suppose $M, N \geq 2$ and let $W$ be the subspace of the $\mathrm{X}(\mathfrak{osp}_{M|N})$-module $(\mathbb{C}^{M|N})^{\otimes(m+1)}$ spanned by the elements $w_k$ for $M+1 \leq k \leq M+N$, and let $W' := W \oplus \mathbb{C}w_{\widehat{m}}$. Then:*

(i) *When $M$ is even, the subspace $W$ is invariant under the operators $T_{ij}(u)$ for $i,j \in \{M+1, \ldots, M+N\}$. Furthermore, the assignment $\dot{T}_{ij}(u) \mapsto T_{\nu(i)\nu(j)}(u)$ where $\nu(i) = M+i$ for $1 \leq i \leq N$, equips $W$ with an $\mathrm{X}(\mathfrak{osp}_{0|N})$-module structure isomorphic to the vector representation $\mathbb{C}^{0|N}$ as in (2.3.8).*

(ii) *When $M$ is odd, the subspace $W'$ is invariant under the operators $T_{ij}(u)$ for $i,j \in \{\widehat{m}, M+1, \ldots, M+N\}$. Furthermore, the assignment $\acute{T}_{ij}(u) \mapsto T_{\nu(i)\nu(j)}(u)$ where $\nu(1) = \widehat{m}$ and $\nu(i) = M-1+i$ for $2 \leq i \leq 1+N$ equips $W'$ with an $\mathrm{X}(\mathfrak{osp}_{1|N})$-module structure isomorphic to the vector representation $\mathbb{C}^{1|N}$ as in (2.3.8).*

*Proof.* (i) Let us set $V = (\mathbb{C}^{M|N})^{\otimes(m+1)}$ and write $\mathcal{F}_M^+$ to denote restriction functor $\mathcal{F}^+$ in Proposition 3.1.9. Applying the composition $\mathcal{F}_2^+\circ\cdots\circ\mathcal{F}_{M-2}^+\circ\mathcal{F}_M^+$ to the representation $V$ yields an $\mathrm{X}(\mathfrak{osp}_{0|N})$-module, denoted $V^{\Sigma_m+}$. By Proposition 3.1.6 and Lemma 3.2.18, the subspace $W$ is contained in $V^{\Sigma_m+}$. We will now prove that the action of $\mathrm{X}(\mathfrak{osp}_{0|N})$ on $W$ is determined by the formula

$$T_{ij}(u)w_k = \delta_{ij}w_k - \frac{\delta_{jk}}{u}w_i + \frac{\delta_{\bar{\imath}k}\theta_i\theta_j}{u+\kappa-m}w_{\bar{\jmath}}$$

for indices $M+1 \le i, j \le M+N$ which will finish the proof. For ease of computations, we shall write $w_k = \sum_{a=1}^{m+1}(-1)^{m+1-a}f_k^{[a]} = \sum_{a=0}^{m}(-1)^a f_k^{[m+1-a]}$ where

$$f_k^{[a]} := \sum_{\sigma\in\mathfrak{S}_m}(\operatorname{sgn}\sigma)e_{\sigma(1)}\otimes\cdots\otimes e_{\sigma(a-1)}\otimes e_k\otimes e_{\sigma(a)}\otimes\cdots\otimes e_{\sigma(m)}.$$

Assuming $M+1 \le i, j \le M+N$, one can compute the following for $0 \le a \le m$:

$$\begin{aligned}T_{ij}(u)f_k^{[m+1-a]} &= \delta_{ij}f_k^{[m+1-a]} - \frac{\delta_{jk}}{u-a}f_i^{[m+1-a]} + \frac{\delta_{jk}(-1)^{[a]}}{(u-a-1)(u-a)}\sum_{s=a+1}^{m}(-1)^s f_i^{[m+1-s]}\\ &+ \frac{\delta_{\bar{\imath}k}\theta_i\theta_j}{u+\kappa-a}f_{\bar{\jmath}}^{[m+1-a]} + \frac{\delta_{\bar{\imath}k}\theta_i\theta_j(-1)^{[a]}}{(u+\kappa-a)(u+\kappa-a+1)}\sum_{s=1}^{a-1}(-1)^s f_{\bar{\jmath}}^{[m+1-s]},\end{aligned}$$

so either $\delta_{ij}$, $\delta_{jk}$, or $\delta_{\bar{\imath}k}$ occurs in each term of the expression $T_{ij}(u)w_k$. Writing

$$T_{ij}(u)w_k = \sum_{a=0}^{m}(-1)^a T_{ij}(u)f_k^{[m+1-a]},$$

we see that the coefficient of $\delta_{ij}$ in $T_{ij}(u)w_k$ is $w_k$. Furthermore, we find that the coefficient of $f_i^{[m+1-a]}$ in $T_{ij}(u)w_k$ is given by

$$-\delta_{jk}(-1)^a\left(\frac{1}{u-a} - \sum_{s=0}^{a-1}\frac{1}{(u-s-1)(u-s)}\right) = \delta_{jk}(-1)^a u^{-1},$$

so the coefficient of $\delta_{jk}$ in $T_{ij}(u)w_k$ is given by $-\delta_{jk}u^{-1}w_i$. Similarly, we find that the coefficient of $f_{\bar{\jmath}}^{[m+1-a]}$ in $T_{ij}(u)w_k$ is given by

$$\delta_{\bar{\imath}k}\theta_i\theta_j(-1)^a\left(\frac{1}{u+\kappa-a} + \sum_{s=a+1}^{m}\frac{1}{(u+\kappa-s)(u+\kappa-s+1)}\right) = \frac{\delta_{\bar{\imath}k}\theta_i\theta_j(-1)^a}{u+\kappa-m},$$

so the coefficient of $\delta_{\bar{\imath}k}$ in $T_{ij}(u)w_k$ is given by $\delta_{\bar{\imath}k}\theta_i\theta_j(u+\kappa-m)^{-1}w_{\bar{\jmath}}$.

Therefore, the subspace $W$ is invariant under the action of $\mathrm{X}(\mathfrak{osp}_{0|N})$ and there is an $\mathrm{X}(\mathfrak{osp}_{0|N})$-module isomorphism $W \cong \mathbb{C}^{0|N}$ given via the assignment $w_k \mapsto e_{k-M}$, where $\kappa_{0,N} = \kappa_{M,N} - m = \kappa - m$ since $M$ is even.

(ii) Similar to the proof of (i), applying the composition $\mathcal{F}_3^+ \circ \cdots \circ \mathcal{F}_{M-2}^+ \circ \mathcal{F}_M^+$ to the representation $V = (\mathbb{C}^{M|N})^{\otimes(m+1)}$ yields an $\mathrm{X}(\mathfrak{osp}_{1|N})$-module $V^{\Sigma_m +}$. Proposition 3.1.6 and Lemma 3.2.18 shows that the subspace $W'$ is contained in $V^{\Sigma_m +}$ and the action of $\mathrm{X}(\mathfrak{osp}_{1|N})$ on $W'$ can be similarly determined to have the form

$$T_{ij}(u)w_k = \delta_{ij}w_k + \frac{\delta_{jk}(-1)^{[i]}}{u}w_i - \frac{\delta_{\bar{\imath}k}\theta_i\theta_j(-1)^{[i][j]}}{u+\kappa-m}w_{\bar{\jmath}}$$

for indices $i,j \in \{\widehat{m}, M+1, \ldots, M+N\}$. Hence, there is an $\mathrm{X}(\mathfrak{osp}_{1|N})$-module isomorphism $W' \cong \mathbb{C}^{1|N}$ given via the assignment $w_{\widehat{m}} \mapsto e_1$ and $w_k \mapsto e_{k+1-M}$ for $M+1 \le k \le M+N$, where $\kappa_{1,N} = \kappa_{M,N} - m = \kappa - m$ since $M$ is odd. □

For $1 \le d \le n$, we now construct new representations of $\mathrm{X}(\mathfrak{osp}_{M|N})$ by composing the representation $\varrho_{m\to 0}$ (3.2.30) with the shift automorphism $\tau_a$ (2.2.10) for each of the values $a = 0, 1, \ldots, d-1$. Tensoring together these resulting representations gives

$$\chi_{0\to(d-1)} := (\otimes_{i=0}^{d-1} \varrho_{m\to 0} \circ \tau_i) \circ \Delta_{d-1} \colon \mathrm{X}(\mathfrak{osp}_{M|N}) \to \mathrm{End}(\mathbb{C}^{M|N})^{\otimes(m+1)d}. \qquad (3.2.32)$$

Further, for integers $1 \le d \le n$ we consider the element

$$\xi_d = \sum\nolimits_{\sigma \in \widehat{\mathfrak{S}}_d} (\operatorname{sgn}\sigma) w_{\sigma(M+1)} \otimes \cdots \otimes w_{\sigma(M+d)} \in (\mathbb{C}^{M|N})^{\otimes(m+1)d},$$

where $\widehat{\mathfrak{S}}_d$ is the symmetric group on the symbols $\{M+1, \ldots, M+d\}$. Furthermore, we let

$$\dot{\varrho} \colon \mathrm{X}(\mathfrak{osp}_{0|N}) \to \mathrm{End}\, W \quad \text{and} \quad \varrho' \colon \mathrm{X}(\mathfrak{osp}_{1|N}) \to \mathrm{End}\, W'$$

denote the representations as described in Proposition 3.2.19. Similarly, we may compose these representations with their respective the shift automorphisms $\tau_a$ for each of the values $a = 0, 1, \ldots, d-1$. Tensoring together these resulting representations gives $\dot{\varrho}_{0\to(d-1)} := (\otimes_{i=0}^{d-1} \dot{\varrho} \circ \tau_i) \circ \Delta_{d-1}$ and $\varrho'_{0\to(d-1)} := (\otimes_{i=0}^{d-1} \varrho' \circ \tau_i) \circ \Delta_{d-1}$:

$$\dot{\varrho}_{0\to(d-1)} \colon \mathrm{X}(\mathfrak{osp}_{0|N}) \to \mathrm{End}\, W^{\otimes d} \quad \text{and} \quad \varrho'_{0\to(d-1)} \colon \mathrm{X}(\mathfrak{osp}_{1|N}) \to \mathrm{End}(W')^{\otimes d}.$$

**Proposition 3.2.20.** *Suppose $M, N \geq 2$. Given $1 \leq d \leq n$, $\xi_d \in (\mathbb{C}^{M|N})^{\otimes(m+1)d}$, and the action (3.2.32) on the space $(\mathbb{C}^{M|N})^{\otimes(m+1)d}$, the $\mathrm{X}(\mathfrak{osp}_{M|N})$-submodule generated by $\xi_d$ is a highest weight module with highest weight vector $\xi_d$ and highest weight $\mu(u) = (\mu_k(u))_{k=1}^{M+N}$ given by*

$$\mu_k(u) = \begin{cases} \dfrac{u+1}{u-d+1} & \text{if} \quad 1 \leq k \leq m, \\ \dfrac{u+\kappa-m-d}{u+\kappa-m} & \text{if} \quad \widehat{m}+1 \leq k \leq M, \\ \dfrac{u-d}{u-d+1} & \text{if} \quad M+1 \leq k \leq M+d, \\ \dfrac{u-\kappa+m+1}{u-\kappa+m} & \text{if} \quad M+N-d+1 \leq k \leq M+N, \\ 1 & \text{otherwise.} \end{cases} \tag{3.2.33}$$

*The quotient of $\mathrm{X}(\mathfrak{osp}_{M|N})\xi_d$ by its maximal submodule will be isomorphic to $L(\mu(u))$ with $\dim L(\mu(u)) < \infty$. Its first two Drinfel'd polynomials are $\widetilde{Q}(u) = u+1$ and $Q(u) = u-d$.*

*When $d < n$, the remaining Drinfel'd polynomials are $P_{M+d}(u) = u-d+1$ and $P_k(u) = 1$ for all $k \in I \setminus \{M+d\}$. Otherwise when $d = n$, then $P_{M+n}(u) = u-n$ when $M$ is odd or $P_{M+n}(u) = u-n+2$ when $M$ is even, with $P_k(u) = 1$ for all $k \in I \setminus \{M+n\}$.*

*Proof.* For now, we will consider how $T_{ij}(u)$ acts on tensor products $w_{p_1} \otimes \cdots \otimes w_{p_d}$ of the elements (3.2.29) with indices satisfying $M+1 \leq p_1, \ldots, p_d \leq M+d$. Such action is described by the formula

$$\begin{aligned} T_{ij}(u)&(w_{p_1} \otimes \cdots \otimes w_{p_d}) \\ &= \sum_{a_1,\ldots,a_{d-1}=1}^{M+N} (-1)^{(d-1)[j]+\sum_{i=1}^{d-1}[a_i]} T_{ia_1}(u)w_{p_1} \otimes T_{a_1a_2}(u-1)w_{p_2} \otimes \cdots \\ &\qquad \cdots \otimes T_{a_{d-1}j}(u-(d-1))w_{p_d}. \end{aligned} \tag{3.2.34}$$

Using the relations described in Lemma 3.2.18, one can show $T_{ij}(u)(w_{p_1} \otimes \cdots \otimes w_{p_d}) = 0$ for indices $(i,j) \in \Gamma_{0,0} \cup \Gamma_{0,1} \cup \Gamma_{1,0}$ and $T_{kk}(u)(w_{p_1} \otimes \cdots \otimes w_{p_d}) = \lambda_k(u)(w_{p_1} \otimes \cdots \otimes w_{p_d})$ for indices $1 \leq k \leq M$ where $\lambda_k(u)$ is as described in (3.2.33).

Assuming $M$ is even and $M+1 \leq i,j \leq M+N$, Lemma 3.2.18 shows that the indices in (3.2.34) can be restricted to $M+1 \leq a_1, \ldots, a_{d-1}, \leq M+N$. Moreover, since $\xi_d \in W^{\otimes d}$, there is an equality $\chi_{0\to(d-1)}(T_{ij}(u))\xi_d = \dot{\varrho}_{0\to(d-1)}(\dot{T}_{\nu^{-1}(i)\nu^{-1}(j)}(u))\xi_d$. Thus, by the $\mathrm{X}(\mathfrak{osp}_{0|N})$-module isomorphism $\mathbb{C}^{0|N} \cong W$, $e_k \mapsto w_{M+k}$ and the algebra isomorphism $\mathrm{X}(\mathfrak{sp}_N) \cong \mathrm{X}(\mathfrak{osp}_{0|N})$, $\mathsf{T}_{ij}(-u) \mapsto \dot{T}_{ij}(u)$, one can use the properties of the vector representation of $\mathrm{X}(\mathfrak{sp}_N)$ as in [AMR06, Theorem 5.16] to conclude $T_{ij}(u)\xi_d = 0$ for $(i,j) \in \Gamma_{1,1}$ and $T_{kk}(u)\xi_d = \lambda_k(u)\xi_d$ for the remaining indices $M+1 \leq k \leq M+N$.

Similarly, when $M$ is odd and $i,j \in \{\widehat{m}, M+1, \dots, M+N\}$, Lemma 3.2.18 shows that the indices in (3.2.34) can be restricted to $a_1, \dots, a_{d-1} \in \{\widehat{m}, M+1, \dots, M+N\}$. Furthermore, as $\xi_d \in (W')^{\otimes d}$, then $\chi_{0\to(d-1)}(T_{ij}(u))\xi_d = \varrho'_{0\to(d-1)}(\acute{T}_{(\nu')^{-1}(i)(\nu')^{-1}(j)}(u))\xi_d$. Thus, by the $\mathrm{X}(\mathfrak{osp}_{1|N})$-module isomorphism $\mathbb{C}^{1|N} \cong W'$ and the superalgebra isomorphism $\mathrm{X}^{\mathbf{d}}(\mathfrak{osp}_{1|N}) \cong \mathrm{X}(\mathfrak{osp}_{1|N})$ where $\mathbf{d} = \{n+1\}$, one can use the properties of the vector representation of $\mathrm{X}^{\mathbf{d}}(\mathfrak{osp}_{1|N})$ as in [Mol23b, § 3] to conclude $T_{ij}(u)\xi_d = 0$ for $(i,j) \in \Gamma_{1,1}$ and $T_{kk}(u)\xi_d = \lambda_k(u)\xi_d$ for the remaining indices $M+1 \leq k \leq M+N$. □

### 3.2.5 Classification conjectures

In this final subsection, we formulate conjectures for the classifications of the sets $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{X}(\mathfrak{osp}_{M|N}))_{/\sim}$ and $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{Y}(\mathfrak{osp}_{M|N}))_{/\sim}$, which will be stated shortly. We note that the much of the style and argumentation in this subsection mirrors that given in [Wen19, § 4.1]. At the end of the subsection, we will also see examples of infinite-dimensional irreducible representations of $\mathbf{X}(\mathfrak{osp}_{M|N})$ that arise from spinor representations of $\mathfrak{osp}_{M|N}$ which demonstrates how such conjectures do not extend beyond the finite-dimensional setting.

Supposing $M, N \geq 2$, we recall the map

$$\begin{aligned}\mho\colon\ \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{X}(\mathfrak{osp}_{M|N}))_{/\sim} &\to \left\{(B_k(u))_{k=1}^{m+n+1} \in \mathbb{C}[u]^2_{\mathrm{cp,ed}} \times \mathbb{C}[u]^{m+n-1} \,|\, B_k(u) \text{ is monic}\right\}\\ L(\lambda(u)) &\mapsto (\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})\end{aligned}$$

and remember such map is not injective: $\mho(L(\lambda(u))) = \mho(L(\mu(u)))$ if and only if there exists a series $f(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that $\mu(u) = f(u)\lambda(u)$.

In Corollary 3.2.14, it is established that the type I fundamental representation $L(\lambda(u);\alpha,\beta)$ is finite-dimensional if and only if $\alpha - \beta \in O$, where $O$ is a certain non-trivial subset of $\frac{1}{2}\mathbb{Z}^+$. Accordingly, we let $\mathbb{C}[u]^2_{O,\mathrm{ed}}$ denote the subset of $\mathbb{C}[u]^2$ described by pairs of polynomials $(B_1(u), B_2(u))$ that have equal degree and can be written as $B_1(u) = c\prod_{i=1}^n (u-\alpha_i)$ and $B_2(u) = d\prod_{i=1}^n (u-\beta_i)$ such that $c, d \in \mathbb{C}^*$ and

$\alpha_i - \beta_i \in O \subseteq \frac{1}{2}\mathbb{Z}^+$ for each $1 \le i \le n$.

When $M \ge 3$, we conjecture that Theorem 3.2.8 can be refined to establish that the Drinfel'd polynomials $\widetilde{Q}(u)$ and $Q(u)$ satisfy $(\widetilde{Q}(u), Q(u)) \in \mathbb{C}[u]^2_{O,\mathrm{ed}}$. In this case, we assert there exists a well-defined map

$$\begin{aligned}\mho_O\colon \operatorname{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{X}(\mathfrak{osp}_{M|N}))_{/\sim} &\to \left\{(B_k(u))_{k=1}^{m+n+1} \in \mathbb{C}[u]^2_{O,\mathrm{ed}} \times \mathbb{C}[u]^{m+n-1} \,|\, B_k(u) \text{ is monic}\right\}\\ L(\lambda(u)) &\mapsto (\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})\end{aligned}$$

and that $\mho_O(L(\lambda(u))) = \mho_O(L(\mu(u)))$ if and only if there exists $f(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that $\mu(u) = f(u)\lambda(u)$.

**Conjecture 3.2.21.** *The map $\mho_O$ is a surjective function.*

In the following, let $\varphi_{\lambda(u)}$ denote the morphism $\mathbf{X}(\mathfrak{osp}_{M|N}) \to \operatorname{End} L(\lambda(u))$, and recall the central series $\mathcal{Y}(u)$ defined by (2.4.4). We therefore have the following conjecture for the classification for the set $\operatorname{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{X}(\mathfrak{osp}_{M|N}))_{/\sim}$:

**Conjecture 3.2.22.** *Suppose $M \ge 3$, $N \ge 2$. The isomorphism classes of finite-dimensional irreducible representations of the extended Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$ are parametrized by tuples*

$$(f(u); (B_k(u))_{k=1}^{m+n+1}) \in (1+u^{-1}\mathbb{C}[\![u^{-1}]\!]) \times \mathbb{C}[u]^2_{O,\mathrm{ed}} \times \mathbb{C}[u]^{m+n-1},$$

*where the polynomials $(B_k(u))_{k=1}^{m+n+1}$ are monic. The underlying correspondence $\mho_X$ is given by*

$$\mho_X\big(L(\lambda(u))\big) = (f(u); \widetilde{Q}(u), Q(u); (P_k(u))_{k\in I}),$$

*where $f(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ is the unique series such that $\mu_f^*\varphi_{\lambda(u)}(\mathcal{Y}(u)) = \mathrm{id}$ and $(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$ are the Drinfel'd polynomials corresponding to $L(\lambda(u))$ under the map $\mho_O$.*

*Proof.* We first show the map $\mho_X$ is well-defined. Assuming $\dim L(\lambda(u)) < \infty$, the irreducibility of $L(\lambda(u))$ implies that the central series $\mathcal{Y}(u)$ acts by a scalar series $y(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$. However, as shown in the proof of Theorem 2.4.7, there is an equality $\mu_h(\mathcal{Y}(u)) = h(u)\mathcal{Y}(u)$ for all $h(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$; thus, $f(u) = y(u)^{-1}$ is the unique series satisfying $(\varphi_{\lambda(u)} \circ \mu_f)(\mathcal{Y}(u)) = \mathrm{id}$. Furthermore, since $\mu_h^* \varphi_{\lambda(u)} \cong \varphi_{h(u)\lambda(u)}$ for all $h(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, there is the assignment

$$\mho_X \colon L(h(u)\lambda(u)) \mapsto (h(u)^{-1}y(u)^{-1}; \widetilde{Q}(u), Q(u); (P_k(u))_{k\in I}).$$

The surjectivity of $\mho_X$ follows from the surjectivity of $\mho_O$, so we can associate a finite-dimensional representation $L(\lambda(u))$ to any tuple $(\widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$ satisfying the appropriate conditions. In particular, $\mho_X$ maps $L(f(u)^{-1}y(u)^{-1}\lambda(u))$ to $(f(u); \widetilde{Q}(u), Q(u); (P_k(u))_{k\in I})$ for any $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$.

For injectivity, $\mho_X(L(\lambda(u))) = \mho_X(L(\mu(u)))$ infers $\mho_O(L(\lambda(u))) = \mho_O(L(\mu(u)))$, implying $\mu(u) = h(u)\lambda(u)$ for some series $h(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$. However, $h(u)^{-1}y(u)^{-1} = y(u)^{-1}$ if and only if $h(u) = 1$. □

Shifting our attention to the representation theory of $\mathbf{Y}(\mathfrak{osp}_{M|N})$, we note the projection $\varepsilon_Y \colon \mathbf{X}(\mathfrak{osp}_{M|N}) \twoheadrightarrow \mathbf{Y}(\mathfrak{osp}_{M|N})$, $T(u) \mapsto \mathcal{T}(u)$ induces the pullback functor

$$\varepsilon_Y^* \colon \mathrm{Rep}(\mathbf{Y}(\mathfrak{osp}_{M|N})) \to \mathrm{Rep}(\mathbf{X}(\mathfrak{osp}_{M|N})),$$

while the embedding $\iota_Y \colon \mathbf{Y}(\mathfrak{osp}_{M|N}) \hookrightarrow \mathbf{X}(\mathfrak{osp}_{M|N})$, $\mathcal{T}(u) \mapsto \mathcal{Y}(u)^{-1}T(u)$ gives rise to another pullback functor

$$\iota_Y^* \colon \mathrm{Rep}(\mathbf{X}(\mathfrak{osp}_{M|N})) \to \mathrm{Rep}(\mathbf{Y}(\mathfrak{osp}_{M|N})).$$

Since $\varepsilon_Y(\mathcal{Y}(u)) = \mathbf{1}$, one can readily verify $\iota_Y^* \circ \varepsilon_Y^* = \mathbb{1}$. In fact, for those representations $\varphi$ of $\mathbf{X}(\mathfrak{osp}_{M|N})$ satisfying $\varphi(\mathcal{Y}(u)) = \mathrm{id}$, then one can also get $(\varepsilon_Y^* \circ \iota_Y^*)(\varphi) = \varphi$.

As $\varepsilon_Y$ is an epimorphism, $\varepsilon_Y^*$ restricts to $\mathrm{Rep}^{\mathrm{irr}}(\mathbf{Y}(\mathfrak{osp}_{M|N})) \to \mathrm{Rep}^{\mathrm{irr}}(\mathbf{X}(\mathfrak{osp}_{M|N}))$, and consequently, $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{Y}(\mathfrak{osp}_{M|N})) \to \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{X}(\mathfrak{osp}_{M|N}))$. Conversely, elements of the center $\mathbf{ZX}(\mathfrak{osp}_{M|N})$ act on any finite-dimensional irreducible representation $V$ of $\mathbf{X}(\mathfrak{osp}_{M|N})$ by multiplication of non-zero scalars in $\mathbb{C}$. Under the induced action of the Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$ by the embedding, $V$ remains irreducible as a $\mathbf{Y}(\mathfrak{osp}_{M|N})$-module. Hence, $\iota_Y^*$ similarly restricts to $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{X}(\mathfrak{osp}_{M|N})) \to \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{Y}(\mathfrak{osp}_{M|N}))$. In particular, the function

$$\Upsilon \colon \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{Y}(\mathfrak{osp}_{M|N}))_{/\sim} \to \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{X}(\mathfrak{osp}_{M|N}))_{/\sim}, \quad [V] \mapsto [\varepsilon_Y^*(V)]$$

is injective with image equal to

$$\{L(\lambda(u)) \in \mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathbf{X}(\mathfrak{osp}_{M|N}))_{/\sim} \mid \varphi_{\lambda(u)}(\mathcal{Y}(u)) = \mathrm{id}\},$$

where $\varphi_{\lambda(u)}$ denotes the morphism $\mathbf{X}(\mathfrak{osp}_{M|N}) \to \mathrm{End}\, L(\lambda(u))$.

**Conjecture 3.2.23.** *Suppose $M \geq 3$, $N \geq 2$. The isomorphism classes of finite-dimensional irreducible representations of the Yangian $\mathbf{Y}(\mathfrak{osp}_{M|N})$ are parametrized by tuples*

$$(B_k(u))_{k=1}^{m+n+1} \in \mathbb{C}[u]^2_{O,\mathrm{ed}} \times \mathbb{C}[u]^{m+n-1},$$

*where the polynomials $(B_k(u))_{k=1}^{m+n+1}$ are monic. The underlying correspondence is given by $\Omega_X \circ \Upsilon$.*

*Proof.* Under the map $\Omega_X$, the image $\mathrm{im}\,\Upsilon$ is mapped to tuples $(1; (B_k(u))_{k=1}^{m+n+1})$, where the polynomials $(B_k(u))_{k=1}^{m+n+1}$ are monic such that $B_1(u)$ and $B_2(u)$ are coprime of the same polynomial degree. □

**Definition 3.2.24.** The *fundamental representations* of $\mathbf{Y}(\mathfrak{osp}_{M|N})$ are those irreducible representations that correspond to Drinfel'd polynomials of the form

$$\big(u+\alpha, u+\beta; (1)_{k\in I}\big) \quad \text{or} \quad \big(1, 1; (u+\gamma)^{\delta_{ik}})_{k\in I}\big)$$

for $i \in I$ and $\alpha, \beta, \gamma \in \mathbb{C}$ where $\alpha \neq \beta$. The fundamental representations corresponding to the first tuple are called *type* I and denoted $L(\alpha, \beta)$, whereas those corresponding to the second tuple are called *type* II and denoted $L(i : \gamma)$.

**Conjecture 3.2.25.** *Suppose $M \geq 3$, $N \geq 2$. Given any finite-dimensional irreducible $\mathbf{Y}(\mathfrak{osp}_{M|N})$-module $V$, there exists $r \in \mathbb{N}$, $i_1, \ldots, i_r \in I$, and $\alpha, \beta, \gamma_1, \ldots, \gamma_r \in \mathbb{C}$ where $\alpha - \beta \in O$, such that $V$ is isomorphic to the irreducible quotient of*

$$\mathbf{Y}(\mathfrak{osp}_{M|N})(\zeta \otimes \xi_1 \otimes \cdots \otimes \xi_r) \subseteq L(\alpha, \beta) \otimes L(i_1 : \gamma_1) \otimes \cdots \otimes L(i_r : \gamma_r)$$

*where $\zeta \in L(\alpha, \beta)$, $\xi_k \in L(i_k : \gamma_k)$, $1 \leq k \leq r$, are the highest weight vectors.*

We now conclude this subsection by investigating a family of infinite-dimensional irreducible representations of the extended Yangian $\mathbf{X}(\mathfrak{osp}_{M|N})$ and demonstrate how the Drinfel'd polynomial relations fail in these cases. As is convention, we still set $m = \lfloor \frac{M}{2} \rfloor$, $\widehat{m} = \lceil \frac{M}{2} \rceil$, and $n = \frac{N}{2}$.

The superexterior algebra $\Lambda(\mathbb{C}^{m|n})$ on $\mathbb{C}^{m|n}$ is the quotient $T(\mathbb{C}^{m|n})/\mathcal{J}$, where $T(\mathbb{C}^{m|n})$ is the tensor superalgebra on $\mathbb{C}^{m|n}$ and $\mathcal{J}$ is the two-sided graded ideal generated by elements of the form $x \otimes y + (-1)^{[x][y]} y \otimes x$, where $x, y \in \mathbb{C}^{m|n}$ are homogeneous. Letting $\{\zeta_{\widehat{m}+1}, \dots, \zeta_M\} \cup \{y_{M+n+1}, \dots, y_{M+N}\}$ denote a homogeneous bases for $\mathbb{C}^{m|n}$ where $[\zeta_i] = \bar{0}$ for $\widehat{m}+1 \leq i \leq M$ and $[y_k] = \bar{1}$ for $M+n+1 \leq k \leq M+N$, then $\Lambda(\mathbb{C}^{m|n})$ can be regarded as the unital associative $\mathbb{C}$-superalgebra on the even generators $\{\zeta_{\widehat{m}+1}, \dots, \zeta_M\}$ and odd generators $\{y_{M+n+1}, \dots, y_{M+N}\}$ subject to the relations

$$\zeta_i \zeta_j = -\zeta_j \zeta_i, \qquad y_k y_l = y_l y_k, \qquad \text{and} \qquad \zeta_i y_k = -y_k \zeta_i,$$

for all $\widehat{m}+1 \leq i, j \leq M$ and $M+n+1 \leq k, l \leq M+N$. We note that the superexterior algebra $\Lambda(\mathbb{C}^{m|n})$ is finite-dimensional if and only if $n = 0$.

Given a superalgebra $\mathcal{A}$, we say that a homogeneous linear map $D \colon \mathcal{A} \to \mathcal{A}$ is a *graded anti-derivation* if $D(xy) = D(x)y - (-1)^{[D][x]} x D(y)$ for all homogeneous elements $x, y \in \mathcal{A}$. For indices $\widehat{m}+1 \leq i \leq M$ we let $\partial_{\zeta_i} \in \operatorname{End} \Lambda(\mathbb{C}^{m|n})$ denote the even anti-derivation defined by $\partial_{\zeta_i}(\zeta_j) = \delta_{ij}$ and $\partial_{\zeta_i}(y_k) = 0$ for indices $\widehat{m}+1 \leq j \leq M$ and $M+n+1 \leq k \leq M+N$. For indices $M+n+1 \leq k \leq M+N$ we let $\partial_{y_k} \in \operatorname{End} \Lambda(\mathbb{C}^{m|n})$ denote the odd anti-derivation defined by $\partial_{y_k}(\zeta_i) = 0$ and $\partial_{y_k}(y_l) = \delta_{kl}$ for the integers $\widehat{m}+1 \leq i \leq M$ and $M+n+1 \leq k, l \leq M+N$.

Letting $m_{\zeta_i}, m_{y_k} \in \operatorname{End} \Lambda(\mathbb{C}^{m|n})$ denote the left multiplication maps by $\zeta_i$ and $y_k$, respectively, we observe that $m_{\zeta_i}$ will be even and $m_{y_k}$ will be odd. Furthermore, if $\{\,\cdot\,,\,\cdot\,\}$ denotes the graded anti-commutator $\{x, y\} = xy + (-1)^{[x][y]} yx$, where $x$ and $y$ are homogeneous, then there are the following relations in $\operatorname{End} \Lambda(\mathbb{C}^{m|n})$:

$$\begin{gathered}
m_{\zeta_i} m_{\zeta_j} = -m_{\zeta_j} m_{\zeta_i}, \qquad m_{y_k} m_{y_l} = m_{y_l} m_{y_k}, \qquad m_{\zeta_i} m_{y_k} = -m_{y_k} m_{\zeta_i}, \\
\partial_{\zeta_i} \partial_{\zeta_j} = -\partial_{\zeta_j} \partial_{\zeta_i}, \qquad \partial_{y_k} \partial_{y_l} = \partial_{y_l} \partial_{y_k}, \qquad \partial_{\zeta_i} \partial_{y_k} = -\partial_{y_k} \partial_{\zeta_i}, \\
\{\partial_{\zeta_i}, m_{\zeta_j}\} = \delta_{ij} \operatorname{id}, \quad \{\partial_{y_k}, m_{y_l}\} = \delta_{ij} \operatorname{id}, \quad \partial_{\zeta_i} m_{y_k} = -m_{y_k} \partial_{\zeta_i}, \quad \partial_{y_k} m_{\zeta_i} = -m_{\zeta_i} \partial_{y_k}.
\end{gathered}$$

The spinor representation of $\mathfrak{osp}_{M|N}$ is the representation on $\Lambda(\mathbb{C}^{m|n})$, denoted

$$\mathsf{sp} \colon \mathfrak{osp}_{M|N} \to \mathfrak{gl}\big(\Lambda(\mathbb{C}^{m|n})\big), \tag{3.2.35}$$

which for the indices $\widehat{m}+1 \le i,j \le M$ and $M+n+1 \le k,l \le M+N$ is determined by

$$\begin{aligned}
\mathsf{sp}(F_{ij}) &= m_{\zeta_i}\partial_{\zeta_j} - \frac{1}{2}\delta_{ij}\,\mathrm{id}, \qquad \mathsf{sp}(F_{\bar{i}j}) = -\partial_{\zeta_i}\partial_{\zeta_j}, \qquad \mathsf{sp}(F_{i\bar{j}}) = -m_{\zeta_i}m_{\zeta_j},\\
\mathsf{sp}(F_{kl}) &= m_{y_k}\partial_{y_l} + \frac{1}{2}\delta_{kl}\,\mathrm{id}, \qquad \mathsf{sp}(F_{\bar{k}l}) = \partial_{y_k}\partial_{y_l}, \qquad \mathsf{sp}(F_{k\bar{l}}) = -m_{y_k}m_{y_l},\\
\mathsf{sp}(F_{ik}) &= -m_{\zeta_i}\partial_{y_k}, \quad \mathsf{sp}(F_{\bar{i}\bar{k}}) = m_{y_k}\partial_{\zeta_i}, \quad \mathsf{sp}(F_{\bar{i}k}) = \partial_{\zeta_i}\partial_{y_k}, \quad \mathsf{sp}(F_{i\bar{k}}) = m_{\zeta_i}m_{y_k},
\end{aligned}$$

and also

$$\mathsf{sp}(F_{i\widehat{m}}) = \frac{1}{\sqrt{2}}m_{\zeta_i}, \qquad \mathsf{sp}(F_{\widehat{m}j}) = \frac{1}{\sqrt{2}}\partial_{\zeta_j}$$

when $M$ is odd.

When $M$ is odd, the spinor representation is irreducible. However, when $M$ is even, the spinor representation splits into a direct sum $\Lambda(\mathbb{C}^{m|n}) = \Lambda(\mathbb{C}^{m|n})^+ \oplus \Lambda(\mathbb{C}^{m|n})^-$ of two irreducible submodules, where $\Lambda(\mathbb{C}^{m|n})^+$ is the submodule spanned by monomials consisting of an even amount of generators (not to be confused with the submodule spanned by generators of $\mathbb{Z}_2$-grade $\bar{0}$) and $\Lambda(\mathbb{C}^{m|n})^-$ is the submodule spanned by monomials consisting of an odd amount of generators (not to be confused with the submodule spanned by generators of $\mathbb{Z}_2$-grade $\bar{1}$).

**Proposition 3.2.26.** *The spinor representation* (3.2.35) *of* $\mathfrak{osp}_{M|N}$ *lifts to a representation of the extended Yangian* $\mathbf{X}(\mathfrak{osp}_{M|N})$ *via the assignment*

$$\mathbf{X}(\mathfrak{osp}_{M|N}) \to \operatorname{End}\Lambda(\mathbb{C}^{m|n}), \quad T_{ij}(u) \mapsto \delta_{ij}\,\mathrm{id} + (-1)^{[i]}\mathsf{sp}(F_{ij})u^{-1}.$$

*Proof.* One checks the defining relations (2.2.8) directly with use of the identity

$$\sum_{p=1}^{M+N}(-1)^{[p]}\mathsf{sp}(F_{ip})\mathsf{sp}(F_{pj}) = \left(\frac{\kappa}{2}+\frac{1}{4}\right)\delta_{ij}(-1)^{[i]}\,\mathrm{id} + \kappa\,\mathsf{sp}(F_{ij}).$$

□

By the above proposition, we will obtain either one or two irreducible infinite-dimensional representations of $\mathbf{X}(\mathfrak{osp}_{M|N})$ depending on the parity of $M$. However, in each case we will see that the Drinfel'd polynomial relations for $P_{M+n}(u)$ fails. Indeed, when $M$ is odd, $\Lambda(\mathbb{C}^{m|n})$ is a highest weight module over $\mathbf{X}(\mathfrak{osp}_{M|N})$ with highest weight vector 1. However, we observe that

$$\frac{\lambda_m(u)}{\lambda_{M+n}(u)} = \frac{u}{u+\frac{1}{2}},$$

so the Drinfel'd polynomial relation fails for $P_{M+n}(u)$ fails. Assuming now that $M$ is even, the submodule $\Lambda(\mathbb{C}^{m|n})^+$ will be a highest weight module also with highest weight vector 1. However, we similarly deduce

$$\frac{\lambda_{M+n}(u)}{\lambda_{M+n+1}(u)} = \frac{u+\frac{1}{2}}{u-\frac{1}{2}}.$$

Moreover, the submodule $\Lambda(\mathbb{C}^{m|n})^-$ will be a highest weight module over $\mathbf{X}(\mathfrak{osp}_{M|N})$ with highest weight vector $y_{M+n+1}$. However, we can again compute

$$\frac{\lambda_{M+n}(u)}{\lambda_{M+n+1}(u)} = \frac{u+\frac{3}{2}}{u-\frac{3}{2}},$$

showing that the Drinfel'd polynomial relation for $P_{M+n}(u)$ fails for another time.

# Part Two

## The Periplectic Yangian $\mathbf{Y}(\mathfrak{p}_N)$ and Twisted Yangian $\mathbf{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$

# Chapter 4

# Yangians of Strange Lie Superalgebras

The main purpose of Chapter 4 is to adapt and extend many structural results obtained for Yangians of the strange Lie superalgebras of type $Q$, as found in [Naz99], to the case of type $P$.

The outline will be as follows. The first section §4.1 establishes required notation and introduces the presentations of the strange Lie superalgebras $\mathfrak{s}_N = \mathfrak{p}_N, \mathfrak{q}_N$ that will be used throughout this work. In §4.2, the definitions of Yangians $\mathrm{Y}(\mathfrak{s}_N)$ are provided via the *RTT* realization as originally given by Nazarov. The main result of the chapter resides in §4.3, where the PBW-type Theorem for the Yangian of type $P$ is proven. In subsection §4.3.3, the definition of the Yangians $\mathrm{Y}_\hbar(\mathfrak{s}_N)$ are given via the Rees superalgebra formalism and it is shown that they serve as a homogeneous quantization of $\mathfrak{gl}_{N|N}[z]^\vartheta$, which represents the fixed-point subalgebra of $\mathfrak{gl}_{N|N}[z]$ under a suitable involution $\vartheta$.

## 4.1 Strange Lie Superalgebras

In Kac's original classification of simple Lie superalgebras [Kac77], the two families of classical Lie superalgebras which are not basic are known as the simple strange Lie superalgebras of types $P$ and $Q$. Types $P$ and $Q$ each describe several families of "strange" Lie superalgebras, including such simple strange Lie superalgebras. In this work, we consider the most general families in each type, denoted $\mathfrak{p}_N$ and $\mathfrak{q}_N$

for $N \in \mathbb{Z}^+$, respectively. In fact, we will see that $\mathfrak{p}_N$ and $\mathfrak{q}_N$ may be realized as fixed-point Lie sub-superalgebras of $\mathfrak{gl}_{N|N}$ under certain involutions, which prompts the following notation.

For a positive integer $N \in \mathbb{Z}^+$, we define the set $I_N := \{ i \in \mathbb{Z} \setminus \{0\} \mid -N \le i \le N \}$ and redefine the *gradation index*

$$[\cdot]\colon I_N \to \mathbb{Z}_2, \quad i \mapsto [i] \quad \text{where} \quad [i] = \bar{0} \quad \text{and} \quad [-i] = \bar{1} \text{ for } i > 0. \tag{4.1.1}$$

We denote $\mathbb{C}^{N|N}$ to be the vector space $\mathbb{C}^{2N}$ equipped with the $\mathbb{Z}_2$-grading given by $[e_i] = [i]$, where $\mathbf{B} = \{e_i\}_{i \in I_N}$ is the standard ordered basis of $\mathbb{C}^{2N}$ enumerated from $-N$ to $N$ omitting 0. Consequently, the space of $\mathbb{C}$-linear maps $\mathbb{C}^{N|N} \to \mathbb{C}^{N|N}$, denoted $\operatorname{End} \mathbb{C}^{N|N}$, carries a natural $\mathbb{Z}_2$-grading such that $[E_{ij}] = [i] + [j]$, where $\{E_{ij}\}_{i,j \in I_N}$ is the collection of standard matrix units with respect to the basis $\mathbf{B}$.

We now consider two relevant involutory automorphisms of $\mathfrak{gl}_{N|N} = \mathfrak{gl}(\mathbb{C}^{N|N})$, which we will denote $(-)^{\iota^{\mathsf{Q}}}$ and $(-)^{\iota^{\mathsf{P}}}$. The first is given by

$$(-)^{\iota^{\mathsf{Q}}}\colon \mathfrak{gl}_{N|N} \to \mathfrak{gl}_{N|N}, \quad E_{ij} \mapsto E_{ij}^{\iota^{\mathsf{Q}}} := E_{-i,-j}, \tag{4.1.2}$$

whereas the second is defined by

$$(-)^{\iota^{\mathsf{P}}} := -(-)^{\iota^{\mathsf{Q}}} \circ (-)^{st}\colon \mathfrak{gl}_{N|N} \to \mathfrak{gl}_{N|N}, \quad E_{ij} \mapsto E_{ij}^{\iota^{\mathsf{P}}} := -(-1)^{[i][j]+[i]} E_{-j,-i}, \tag{4.1.3}$$

where $(-)^{st}$ is the super-transpose (2.1.7). We observe that the involutory automorphism $(-)^{\iota^{\mathsf{Q}}}$ can also restrict to one on the special linear Lie superalgebra $\mathfrak{sl}_{N|N}$, which we will denote with an identical symbol. We can now define the following.

**Definition 4.1.1.** The *Lie superalgebra* $\mathfrak{p}_N$ *of type P* (or periplectic Lie superalgebra) is the fixed-point Lie sub-superalgebra $\mathfrak{gl}_{N|N}^{\iota^{\mathsf{P}}}$ of $\mathfrak{gl}_{N|N}$ under the involutory automorphism $(-)^{\iota^{\mathsf{P}}}$:

$$\mathfrak{p}_N := \mathfrak{gl}_{N|N}^{\iota^{\mathsf{P}}} = \{X \in \mathfrak{gl}_{N|N} \mid X^{\iota^{\mathsf{P}}} = X\}.$$

Similarly, the *Lie superalgebra* $\mathfrak{q}_N$ *of type Q* is the fixed-point Lie sub-superalgebra $\mathfrak{sl}_{N|N}^{\iota^{\mathsf{Q}}}$ of $\mathfrak{sl}_{N|N}$ under the involutory automorphism $(-)^{\iota^{\mathsf{Q}}}$:

$$\mathfrak{q}_N := \mathfrak{sl}_{N|N}^{\iota^{\mathsf{Q}}} = \{X \in \mathfrak{sl}_{N|N} \mid X^{\iota^{\mathsf{Q}}} = X\}.$$

Under the identification $\operatorname{End}\mathbb{C}^{N|N} \cong \operatorname{Mat}_{N|N}(\mathbb{C})$, any element $A \in \mathfrak{gl}_{N|N}$ can be identified with a $2 \times 2$ block matrix

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

where $A_{ij} \in \operatorname{Mat}_N(\mathbb{C})$ for all $i, j \in \{0, 1\}$. In particular, we find that $A \in \mathfrak{p}_N$ if and only if $A_{11} = -A_{00}^t$, $A_{01} = A_{01}^t$, and $A_{10} = -A_{10}^t$, where $(-)^t$ denotes the transpose on $\operatorname{Mat}_N(\mathbb{C})$ with respect to the anti-diagonal. We note that the simple Lie superalgebra of type $P$ is the Lie sub-superalgebra of $\mathfrak{p}_N$ consisting of all matrices $A$ with $\operatorname{tr}(A_{00}) = 0 = \operatorname{tr}(A_{11})$.

Similarly, $A \in \mathfrak{q}_N$ if and only if $A_{00} = A_{11}$ and $A_{01} = A_{10}$. One can also consider the Lie sub-superalgebra $\mathfrak{sq}_N := [\mathfrak{q}_N, \mathfrak{q}_N]$ that consists of all matrices $A \in \mathfrak{q}_N$ with $\operatorname{tr}(A_{01}) = \operatorname{tr}(A_{10}) = 0$. Observing that the identity matrix $I$ still lies in $\mathfrak{sq}_N$, the simple Lie superalgebra of type $Q$ is the quotient $\mathfrak{psq}_N := \mathfrak{sq}_N/\mathbb{C}I$.

Returning out attention to the Lie superalgebras $\mathfrak{p}_N$ and $\mathfrak{q}_N$, we wish to express these Lie superalgebras in terms of generators and relations. To this end, we find that $\mathfrak{p}_N$ is generated by the operators

$$\mathsf{E}_{ij} := E_{ij} + E_{ij}^{\iota^{\mathsf{P}}} = E_{ij} - (-1)^{[i][j]+[i]} E_{-j,-i} \in \mathfrak{gl}_{N|N} \quad \text{for all} \quad i, j \in I_N \tag{4.1.4}$$

subject only to the relations

$$\begin{aligned} \left[\mathsf{E}_{ij}, \mathsf{E}_{kl}\right] = \delta_{jk}\mathsf{E}_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\,\mathsf{E}_{kj} \\ - \delta_{i,-k}(-1)^{[i][j]+[i]}\,\mathsf{E}_{-j,l} + \delta_{j,-l}(-1)^{([i]+[j])[k]}\,\mathsf{E}_{k,-i} \\ \text{and} \quad \mathsf{E}_{ij} + (-1)^{[i][j]+[i]}\,\mathsf{E}_{-j,-i} = 0. \end{aligned}$$

Similarly, the Lie superalgebra $\mathfrak{q}_N$ is generated by the operators

$$\mathsf{F}_{ij} := E_{ij} + E_{ij}^{\iota^{\mathsf{Q}}} = E_{ij} + E_{-i,-j} \in \mathfrak{sl}_{N|N} \quad \text{for all} \quad i, j \in I_N \tag{4.1.5}$$

subject only to the relations

$$\begin{aligned} \left[\mathsf{F}_{ij}, \mathsf{F}_{kl}\right] = \delta_{jk}\mathsf{F}_{il} - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\,\mathsf{F}_{kj} + \delta_{j,-k}\mathsf{F}_{-i,l} - \delta_{i,-l}(-1)^{([i]+[j])([k]+[l])}\,\mathsf{F}_{k,-j} \\ \text{and} \quad \mathsf{F}_{ij} - \mathsf{F}_{-j,-i} = 0. \end{aligned}$$

## 4.2 Yangians of Types $P$ and $Q$

In this section, we shall provide the definitions for the Yangians $\mathrm{Y}(\mathfrak{p}_N)$ and $\mathrm{Y}(\mathfrak{q}_N)$ using the *RTT* formalism as first provided by Nazarov in [Naz92].

### 4.2.1 Yangians of strange Lie superalgebras

To start, we recall the *super permutation operator* in $\left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}$ as given by

$$P := \sum_{i,j\in I_N} (-1)^{[j]} E_{ij} \otimes E_{ji}. \tag{4.2.1}$$

By setting $(-)^{\iota_1^K} = (-)^{\iota^K} \otimes \mathrm{id}$, $(-)^{\iota_2^K} = \mathrm{id} \otimes (-)^{\iota^K} \in \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}$ for $K = \mathsf{P}, \mathsf{Q}$, these maps act on the super permutation operator via

$$P^{\iota_2^K} = -P^{\iota_1^K} \quad \text{and} \quad (P^{\iota_1^K})^{\iota_2^K} = (P^{\iota_2^K})^{\iota_1^K} = -P.$$

Defining $Q^K := P^{\iota_2^K}$, so

$$Q^{\mathsf{P}} = -\sum_{i,j\in I_N} (-1)^{[i][j]} E_{ij} \otimes E_{-i,-j} \quad \text{and} \quad Q^{\mathsf{Q}} = \sum_{i,j\in I_N} (-1)^{[j]} E_{ij} \otimes E_{-j,-i}, \tag{4.2.2}$$

the *R-matrix* $R^K(u,v)$ is the rational function in formal parameters $u, v$ taking coefficients in $\left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}$ given by

$$R^K(u,v) := \mathrm{id}^{\otimes 2} - \frac{P}{u-v} - \frac{Q^K}{u+v} \in \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}(u,v). \tag{4.2.3}$$

We recall that for indices $1 \le k < l \le m$, there is a morphism of superalgebras

$$\begin{aligned} \varphi_{kl} \colon \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2} &\to \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes m} \\ a \otimes b &\mapsto \mathbf{1}^{\otimes(k-1)} \otimes a \otimes \mathbf{1}^{\otimes(l-k-1)} \otimes b \otimes \mathbf{1}^{\otimes(m-l)} \end{aligned}$$

and set $X_{kl} = \varphi_{kl}(X)$ for an element $X \in \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}$. When $X = X(u,v)$ depends on formal parameters $u, v$, then we write $X_{kl}(u,v)$ for $\varphi_{kl}(X(u,v))$. In particular, the $R$-matrix (4.2.3) satisfies the *super quantum Yang-Baxter equation* (SQYBE):

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v). \tag{4.2.4}$$

Regarding $(-)^{\iota_1^K}$ and $(-)^{\iota_2^K}$ as maps lifted to act on the space $\left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2}(u,v)$, such act on the $R$-matrix via the formulas

$$\begin{aligned} R^{\mathsf{P}}(u,v)^{\iota_1^{\mathsf{P}}} &= -R^{\mathsf{P}}(u,-v), \quad R^{\mathsf{P}}(u,v)^{\iota_2^{\mathsf{P}}} = -R^{\mathsf{P}}(-u,v), \\ \text{and} \quad R^{\mathsf{Q}}(u,v)^{\iota_1^{\mathsf{Q}}} &= R^{\mathsf{Q}}(-u,v), \quad R^{\mathsf{Q}}(u,v)^{\iota_2^{\mathsf{Q}}} = R^{\mathsf{P}}(u,-v). \end{aligned}$$

Moreover, given the operator $\mathrm{jd} := \sum_{j\in I_N}(-1)^{[j]}E_{j,-j}$, the array of equalities

$$\begin{aligned} P^2 = \mathrm{id}^{\otimes 2}, \quad PQ^{\mathsf{P}} = Q^{\mathsf{P}}, \quad Q^{\mathsf{P}}P = -Q^{\mathsf{P}}, \quad (Q^{\mathsf{P}})^2 = 0, \\ PQ^{\mathsf{Q}} = -\,\mathrm{jd}^{\otimes 2}, \quad Q^{\mathsf{Q}}P = \mathrm{jd}^{\otimes 2}, \quad \text{and} \quad (Q^{\mathsf{Q}})^2 = \mathrm{id}^{\otimes 2}, \end{aligned}$$

infer that the $R$-matrix $R^K(u,v)$ satisfies the properties

$$(R^K(u,v)^{\iota_1^K})^{\iota_2^K} = R^K(-u,-v), \tag{4.2.5}$$

$$R^K(u,v)R^K(-u,-v) = \left(1 - \frac{1}{(u-v)^2} - \frac{\delta_{K\mathsf{Q}}}{(u+v)^2}\right)\mathrm{id}^{\otimes 2}, \tag{4.2.6}$$

known as *crossing symmetry* and *unitarity*, respectively.

Given a superalgebra $\mathcal{A}$ and indices $1 \le k \le m$, we also recall there is a morphism of superalgebras

$$\varphi_k\colon \operatorname{End}(\mathbb{C}^{N|N}) \otimes \mathcal{A} \to \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes m} \otimes \mathcal{A}, \quad \psi \otimes w \mapsto \mathrm{id}^{\otimes(k-1)} \otimes \psi \otimes \mathrm{id}^{\otimes(m-k)} \otimes w,$$

and set $X_k = \varphi_k(X)$ for an element $X \in \operatorname{End}\mathbb{C}^{N|N} \otimes \mathcal{A}$. When $X = X(u)$ depends on a formal parameter $u$, we shall write $X_k(u)$ for the element $\varphi_k(X(u))$. We can now provide the definitions of the Yangians for both types $P$ and $Q$:

**Definition 4.2.1.** The *Yangian* $\mathrm{Y}(\mathfrak{p}_N)$ of $\mathfrak{p}_N$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{\mathcal{T}_{ij}^{(n)} \mid i,j \in I_N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\left[\mathcal{T}_{ij}^{(n)}\right] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the *RTT-relation*

$$\begin{aligned} R^{\mathsf{P}}(u,v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)R^{\mathsf{P}}(u,v) \\ \text{in} \quad \left(\operatorname{End}\mathbb{C}^{N|N}\right)^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{p}_N)[\![u^{\pm 1}, v^{\pm 1}]\!], \end{aligned} \tag{4.2.7}$$

where $R^{\mathsf{P}}(u,v)$ is identified with $R^{\mathsf{P}}(u,v) \otimes \mathbf{1}$, and

$$\mathcal{T}^{\jmath^{\mathsf{P}}}(u)\mathcal{T}(-u) = \mathbb{1} \quad \text{in} \quad \operatorname{End}(\mathbb{C}^{N|N}) \otimes \mathrm{Y}(\mathfrak{p}_N)[\![u^{\pm 1}]\!], \tag{4.2.8}$$

given $\mathcal{T}(u) := \sum_{i,j\in I_N}(-1)^{[i][j]+[j]}E_{ij}\otimes\mathcal{T}_{ij}(u)\in\mathrm{End}(\mathbb{C}^{N|N})\otimes\mathrm{Y}(\mathfrak{p}_N)[\![u^{-1}]\!]$ is the matrix consisting of the series $\mathcal{T}_{ij}(u) := \delta_{ij}\mathbf{1}+\sum_{n=1}^{\infty}\mathcal{T}_{ij}^{(n)}u^{-n}\in\mathrm{Y}(\mathfrak{p}_N)[\![u^{-1}]\!]$ for $i,j\in I_N$ and $\mathcal{T}^{\jmath^{\mathsf{P}}}(u) = ((-)^{\jmath^{\mathsf{P}}}\otimes\mathrm{id})\mathcal{T}(u)$ where $(-)^{\jmath^{\mathsf{P}}} := -(-)^{\iota^{\mathsf{P}}}$.

**Definition 4.2.2.** The *Yangian* $\mathrm{Y}(\mathfrak{q}_N)$ of $\mathfrak{q}_N$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{\mathcal{T}_{ij}^{(n)} \mid i,j\in I_N,\ n\in\mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\left[\mathcal{T}_{ij}^{(n)}\right] := [i]+[j]$ for all $n\in\mathbb{Z}^+$, subject to the *RTT-relation*

$$\begin{aligned} R^{\mathsf{Q}}(u,v)\mathcal{T}_1(u)\mathcal{T}_2(v) &= \mathcal{T}_2(v)\mathcal{T}_1(u)R^{\mathsf{Q}}(u,v) \\ \text{in}\quad \left(\mathrm{End}\,\mathbb{C}^{N|N}\right)^{\otimes 2} &\otimes \mathrm{Y}(\mathfrak{q}_N)[\![u^{\pm1},v^{\pm1}]\!], \end{aligned} \tag{4.2.9}$$

where $R^{\mathsf{Q}}(u,v)$ is identified with $R^{\mathsf{Q}}(u,v)\otimes\mathbf{1}$, and

$$\mathcal{T}^{\iota^{\mathsf{Q}}}(u) = \mathcal{T}(-u) \quad\text{in}\quad \mathrm{End}(\mathbb{C}^{N|N})\otimes\mathrm{Y}(\mathfrak{q}_N)[\![u^{\pm1}]\!], \tag{4.2.10}$$

given $\mathcal{T}(u) := \sum_{i,j\in I_N}(-1)^{[i][j]+[j]}E_{ij}\otimes\mathcal{T}_{ij}(u)\in\mathrm{End}(\mathbb{C}^{N|N})\otimes\mathrm{Y}(\mathfrak{q}_N)[\![u^{-1}]\!]$ is the matrix consisting of the series $\mathcal{T}_{ij}(u) := \delta_{ij}\mathbf{1}+\sum_{n=1}^{\infty}\mathcal{T}_{ij}^{(n)}u^{-n}\in\mathrm{Y}(\mathfrak{q}_N)[\![u^{-1}]\!]$ for $i,j\in I_N$ and $\mathcal{T}^{\iota^{\mathsf{Q}}}(u) = ((-)^{\iota^{\mathsf{Q}}}\otimes\mathrm{id})\mathcal{T}(u)$.

The remainder of this subsection will dedicated to the overview of many structural properties of the Yangian $\mathrm{Y}(\mathfrak{q}_N)$ that have been established in [Naz92], [Naz99]. The treatment of the type $P$ Yangians will be investigated in the following subsections.

In terms of formal power series, the $RTT$-relation (4.2.9) equivalently takes the form

$$\begin{aligned} (-1)^{[i][j]+[i][k]+[j][k]}\left[\mathcal{T}_{ij}(u),\mathcal{T}_{kl}(v)\right] &= \frac{1}{u-v}\big(\mathcal{T}_{kj}(u)\mathcal{T}_{il}(v)-\mathcal{T}_{kj}(v)\mathcal{T}_{il}(u)\big) \\ &\quad -\frac{1}{u+v}\Big((-1)^{[j]+[k]}\mathcal{T}_{-k,j}(u)\mathcal{T}_{-i,l}(v)-(-1)^{[i]+[l]}\mathcal{T}_{k,-j}(v)\mathcal{T}_{i,-l}(u)\Big) \end{aligned} \tag{4.2.11}$$

for all $i,j,k,l\in I_N$, where the above equality may be regarded as one in the extension $\mathrm{Y}(\mathfrak{q}_N)[\![u^{\pm1},v^{\pm1}]\!]$ and $[\,\cdot\,,\cdot\,]$ is understood as the super-bracket

$$\left[\mathcal{T}_{ij}(u),\mathcal{T}_{kl}(v)\right] = \mathcal{T}_{ij}(u)\mathcal{T}_{kl}(v)-(-1)^{([i]+[j])([k]+[l])}\mathcal{T}_{kl}(v)\mathcal{T}_{ij}(u).$$

Furthermore, the relation (4.2.10) is equivalent to the condition that

$$\mathcal{T}_{-i,-j}(u) = (-1)^{[i]+[j]}\mathcal{T}_{ij}(-u) \quad\text{for all}\quad i,j\in I_N, \tag{4.2.12}$$

which is an equality in the space $\mathrm{Y}(\mathfrak{q}_N)[\![u^{-1}]\!]$.

Given any formal series $f = f(u) = 1 + \sum_{n=1}^{\infty} f_n u^{-n} \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, one can readily verify that the mapping

$$\mu_f \colon \mathcal{T}(u) \mapsto f(u)\mathcal{T}(u)$$

defines a superalgebra automorphism of $\mathrm{Y}(\mathfrak{q}_N)$. Furthermore, the Yangian $\mathrm{Y}(\mathfrak{q}_N)$ admits at least two important superalgebra anti-automorphisms. First, by regarding $T(u)$ as a formal power series in $u^{-1}$ whose coefficients lie in $\mathrm{End}(\mathbb{C}^{N|N}) \otimes \mathrm{Y}(\mathfrak{q}_N)$, its constant term is the unit object $\mathbb{1} = \mathrm{id} \otimes \mathbf{1}$; hence, $T(u)$ has an inverse $T(u)^{-1}$. In particular, the assignments

$$\begin{aligned} \varsigma \colon \mathcal{T}(u) &\mapsto \mathcal{T}(-u), \\ S \colon \mathcal{T}(u) &\mapsto \mathcal{T}(u)^{-1} \end{aligned}$$

define superalgebra anti-automorphisms of $\mathrm{Y}(\mathfrak{q}_N)$, c.f. [Naz99, §2], [Mol07, Proposition 1.3.3]. The Yangian $\mathrm{Y}(\mathfrak{q}_N)$ comes equipped with a Hopf superalgebra structure as provided by the comultiplication

$$\Delta \colon \; \mathrm{Y}(\mathfrak{q}_N) \to \mathrm{Y}(\mathfrak{q}_N) \otimes \mathrm{Y}(\mathfrak{q}_N), \quad \mathcal{T}(u) \mapsto \mathcal{T}_{[1]}(u)\mathcal{T}_{[2]}(u),$$

the counit

$$\varepsilon \colon \; \mathrm{Y}(\mathfrak{p}_N) \to \mathbb{C}, \quad \mathcal{T}(u) \mapsto \mathbb{1},$$

and the antipode

$$S \colon \; \mathrm{Y}(\mathfrak{q}_N) \to \mathrm{Y}(\mathfrak{q}_N), \quad \mathcal{T}(u) \mapsto \mathcal{T}(u)^{-1}.$$

The Yangians of type $Q$ benefit from the existence of a Hopf superalgebra epimorphism

$$\pi \colon \; \mathrm{Y}(\mathfrak{q}_N) \twoheadrightarrow \mathfrak{U}(\mathfrak{q}_N), \quad \mathcal{T}_{ij}(u) \mapsto \delta_{ij} - (-1)^{[i][j]}\mathsf{F}_{ji}u^{-1} \quad \text{for all} \quad i,j \in I_N,$$

which is referred to as the *evaluation homomorphism*. Hence, there is a pullback functor $\pi^* \colon \mathrm{Rep}(\mathfrak{U}(\mathfrak{q}_N)) \to \mathrm{Rep}(\mathrm{Y}(\mathfrak{q}_N))$, which restricts to $\mathrm{Rep}^{\mathrm{irr}}(\mathfrak{U}(\mathfrak{q}_N)) \to \mathrm{Rep}^{\mathrm{irr}}(\mathrm{Y}(\mathfrak{q}_N))$ by virtue that $\pi$ is surjective. Moreover, composing the map $\pi$ with the Hopf superalgebra morphism

$$\iota\colon \mathfrak{U}(\mathfrak{q}_N) \to \mathrm{Y}(\mathfrak{q}_N), \quad \mathsf{F}_{ij} \mapsto -(-1)^{[i][j]}\mathcal{T}_{ji}^{(1)} \quad \text{for all} \quad i,j \in I_N,$$

gives $\pi \circ \iota = \mathrm{id}$, so $\iota$ is in fact an embedding of Hopf superalgebras.

There are two relevant ascending algebra filtrations on $\mathrm{Y}(\mathfrak{q}_N)$, which we denote $\mathbf{F}(\mathrm{Y}(\mathfrak{q}_N)) = \{\mathbf{F}_n(\mathrm{Y}(\mathfrak{q}_N))\}_{n\in\mathbb{N}}$ and $\mathbf{F}'(\mathrm{Y}(\mathfrak{q}_N)) = \{\mathbf{F}'_n(\mathrm{Y}(\mathfrak{q}_N))\}_{n\in\mathbb{N}}$, given by the respective filtration degree assignments

$$\deg_{\mathbf{F}} \mathcal{T}_{ij}^{(n)} = n-1 \quad \text{and} \quad \deg_{\mathbf{F}'} \mathcal{T}_{ij}^{(n)} = n.$$

for all $i,j \in I_N$ and $n \in \mathbb{Z}^+$. From the defining relations of they Yangian of type $Q$, one can deduce that the associated graded superalgebra induced by the second filtration $\mathbf{F}'(\mathrm{Y}(\mathfrak{q}_N))$ is supercommutative. The more important filtration is the first, which will induce a more interesting associated graded superalgebra:

$$\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N) := \mathrm{gr}_{\mathbf{F}}\,\mathrm{Y}(\mathfrak{q}_N) = \bigoplus_{n\in\mathbb{N}} \mathbf{F}_n(\mathrm{Y}(\mathfrak{q}_N))/\mathbf{F}_{n-1}(\mathrm{Y}(\mathfrak{q}_N)).$$

We note that $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N)$ inherits a $\mathbb{Z}_2$-graded structure from $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N)$ by assigning $\mathbb{Z}_2$-grade $[i]+[j]$ to the image $\overline{\mathcal{T}}_{ij}^{(n)}$ of $\mathcal{T}_{ij}^{(n)}$ in $\mathbf{F}_{n-1}(\mathrm{Y}(\mathfrak{q}_N))/\mathbf{F}_{n-2}(\mathrm{Y}(\mathfrak{q}_N))$. Furthermore, $\mathbf{F}(\mathrm{Y}(\mathfrak{q}_N))$ is a Hopf filtration, so $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N)$ is equipped with an $\mathbb{N}$-graded Hopf superalgebra structure given by the comultiplication

$$\begin{aligned} \mathrm{gr}\,\Delta\colon\ \mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N) &\to \mathrm{gr}\big(\mathrm{Y}(\mathfrak{q}_N)^{\otimes 2}\big) \cong (\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N))^{\otimes 2} \\ \overline{\mathcal{T}}_{ij}^{(n)} &\mapsto \overline{\mathcal{T}}_{ij}^{(n)} \otimes \mathbf{1} + \mathbf{1} \otimes \overline{\mathcal{T}}_{ij}^{(n)}, \end{aligned}$$

the counit

$$\mathrm{gr}\,\varepsilon\colon\ \mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N) \to \mathbb{C}, \quad \overline{\mathcal{T}}_{ij}^{(n)} \mapsto 0,$$

and antipode

$$\mathrm{gr}\,S\colon\ \mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N) \to \mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N), \quad \overline{\mathcal{T}}_{ij}^{(n)} \mapsto -\overline{\mathcal{T}}_{ij}^{(n)},$$

for all $i,j \in I_N$ and $n \in \mathbb{Z}^+$.

Letting $\mathfrak{sl}_{N|N}[z] = \mathfrak{sl}_{N|N} \otimes \mathbb{C}[z]$ denote the polynomial current Lie superalgebra associated to $\mathfrak{sl}_{N|N}$, the involution $(-)^{\iota^{\mathsf{Q}}}$ may be extended to an involutory automorphism of $\mathfrak{sl}_{N|N}[z]$, which we also denote $(-)^{\iota^{\mathsf{Q}}}$, by assigning

$$(X \otimes f(z))^{\iota^{\mathsf{Q}}} = X^{\iota^{\mathsf{Q}}} \otimes f(-z) \quad \text{for all} \quad X \in \mathfrak{sl}_{N|N},\ f(z) \in \mathbb{C}[z].$$

Hence, we define the *twisted current Lie superalgebra* $\mathfrak{sl}_{N|N}[z]^{\iota^{\mathsf{Q}}}$ to be the fixed-point Lie sub-superalgebra of $\mathfrak{sl}_{N|N}[z]$ under the the involutive automorphism $(-)^{\iota^{\mathsf{Q}}}$:

$$\mathfrak{sl}_{N|N}[z]^{\iota^{\mathsf{Q}}} := \left\{ g(z) \in \mathfrak{sl}_{N|N}[z] \mid g(z)^{\iota^{\mathsf{Q}}} = g(z) \right\} = \mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{Q}}}.$$

Using the identification $Xz^n = X \otimes z^n$ for elements in $\mathfrak{gl}_{N|N}[z]$, we find that $\mathfrak{sl}_{N|N}[z]^{\iota^{\mathsf{Q}}}$ is generated by the operators

$$\mathsf{F}_{ij}^{(n)}(z) := E_{ij} z^n + E_{ij}^{\iota^{\mathsf{Q}}} (-z)^n = (E_{ij} + (-1)^n E_{-i,-j}) z^n \in \mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{Q}}} \tag{4.2.13}$$

for all $i, j \in I_N$ and $n \in \mathbb{N}$, subject only to the relations

$$\begin{aligned}\left[\mathsf{F}_{ij}^{(m)}(z), \mathsf{F}_{kl}^{(n)}(z)\right] = \delta_{jk} \mathsf{F}_{il}^{(m+n)}(z) - \delta_{il}(-1)^{([i]+[j])([k]+[l])} \mathsf{F}_{kj}^{(m+n)}(z) \\ + \delta_{j,-k}(-1)^m \mathsf{F}_{-i,l}^{(m+n)}(z) - \delta_{i,-l}(-1)^{([i]+[j])([k]+[l])+m} \mathsf{F}_{k,-j}^{(m+n)}(z)\end{aligned} \tag{4.2.14}$$

and

$$\mathsf{F}_{ij}^{(n)}(z) - (-1)^n \mathsf{F}_{-i,-j}^{(n)}(z) = 0. \tag{4.2.15}$$

In particular, we have the following theorem:

**Theorem 4.2.3** (Theorem 2.3 in [Naz99]). *There is an* $\mathbb{N}$*-graded Hopf superalgebra isomorphism*

$$\Phi \colon \mathfrak{U}(\mathfrak{sl}_{N|N}[z]^{\iota^{\mathsf{Q}}}) \xrightarrow{\sim} \operatorname{gr} \mathrm{Y}(\mathfrak{q}_N), \quad \mathsf{F}_{ij}^{(n-1)} \mapsto -(-1)^{[i][j]} \overline{\mathcal{T}}_{ji}^{(n)} \tag{4.2.16}$$

*for* $i, j \in I_N$, $n \in \mathbb{Z}^+$.

### 4.2.2 The extended Yangian of type $P$

In this subsection, we introduce the extended Yangian of $\mathfrak{p}_N$. In subsequent subsections, we will show that the Yangian $\mathrm{Y}(\mathfrak{p}_N)$ may be regarded as a certain quotient of the extended Yangian by an ideal generated by an infinite number of central elements.

**Definition 4.2.4.** The *extended Yangian* $\mathrm{X}(\mathfrak{p}_N)$ of $\mathfrak{p}_N$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{T_{ij}^{(n)} \mid i,j \in I_N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\big[T_{ij}^{(n)}\big] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the *RTT-relation*

$$\begin{aligned} R^{\mathsf{P}}(u,v)T_1(u)T_2(v) &= T_2(v)T_1(u)R^{\mathsf{P}}(u,v), \\ \text{in} \quad &\big(\operatorname{End}\mathbb{C}^{N|N}\big)^{\otimes 2} \otimes \mathrm{X}(\mathfrak{p}_N)[\![u^{\pm 1}, v^{\pm 1}]\!], \end{aligned} \tag{4.2.17}$$

where $T(u) := \sum_{i,j\in I_N} (-1)^{[i][j]+[j]} E_{ij} \otimes T_{ij}(u) \in \operatorname{End}(\mathbb{C}^{N|N}) \otimes \mathrm{X}(\mathfrak{p}_N)[\![u^{-1}]\!]$ is the matrix consisting of the series $T_{ij}(u) := \delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} T_{ij}^{(n)} u^{-n} \in \mathrm{X}(\mathfrak{p}_N)[\![u^{-1}]\!]$ for $i,j \in I_N$, and $R^{\mathsf{P}}(u,v)$ is identified with $R^{\mathsf{P}}(u,v) \otimes \mathbf{1}$.

In terms of formal power series, the $RTT$-relation (4.2.17) is equivalent to the relations

$$\begin{aligned} \big[T_{ij}(u), T_{kl}(v)\big] = {} & \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\big) \\ & - \frac{1}{u+v}\Big(\delta_{i,-k} \sum_{p\in I_N} (-1)^{[i][j]+[j][p]+[p]} T_{pj}(u)T_{-p,l}(v) \\ & \qquad - \delta_{j,-l} \sum_{p\in I_N} (-1)^{[i][k]+[i]+[j][k]+[j]+[i][p]} T_{k,-p}(v)T_{ip}(u)\Big) \end{aligned} \tag{4.2.18}$$

for all $i,j,k,l \in I_N$, where $[\,\cdot\,,\cdot\,]$ is understood as the Lie superbracket

$$\big[T_{ij}(u), T_{kl}(v)\big] = T_{ij}(u)T_{kl}(v) - (-1)^{([i]+[j])([k]+[l])} T_{kl}(v)T_{ij}(u).$$

For any formal series $f(u) = 1 + \sum_{n=1}^{\infty} f_n u^{-n} \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, the map

$$\mu_f \colon T(u) \mapsto f(u)T(u) \tag{4.2.19}$$

defines an automorphism of $\mathrm{X}(\mathfrak{p}_N)$. Furthermore, by defining $T^{\jmath^{\mathsf{P}}}(u) := ((-)^{\jmath^{\mathsf{P}}} \otimes \mathrm{id})T(u)$ where $(-)^{\jmath^{\mathsf{P}}} = -(-)^{\iota^{\mathsf{P}}}$, so that $T_{ij}^{\jmath^{\mathsf{P}}}(u) := (-1)^{[i][j]+[i]} T_{-j,-i}(u)$, the assignment

$$\jmath^{\mathsf{P}} \colon T(u) \mapsto T^{\jmath^{\mathsf{P}}}(u) \tag{4.2.20}$$

also defines a superalgebra automorphism of $\mathrm{X}(\mathfrak{p}_N)$. We remark that although the map $(-)^{\jmath^{\mathsf{P}}} \colon \operatorname{End}\mathbb{C}^{N|N} \to \operatorname{End}\mathbb{C}^{N|N}$ is an *anti*-automorphism, the induced map $\jmath^{\mathsf{P}}$ on $\mathrm{X}(\mathfrak{p}_N)$ will only be an automorphism due to the crossing symmetry property (4.2.5) of the

$R$-matrix $R^{\mathsf{P}}(u,v)$.

The extended Yangian $\mathrm{X}(\mathfrak{p}_N)$ also has at least two important superalgebra anti-automorphisms. Since $T(u)$ as a formal power series in $\mathrm{End}(\mathbb{C}^{N|N})\otimes\mathrm{X}(\mathfrak{p}_N)[\![u^{-1}]\!]$ whose constant term is the unit object $\mathbb{1}=\mathrm{id}\otimes\mathbf{1}$, it must have an inverse $T(u)^{-1}$. Hence, each mapping

$$\varsigma\colon T(u)\mapsto T(-u), \tag{4.2.21}$$

$$S\colon T(u)\mapsto T(u)^{-1} \tag{4.2.22}$$

induces a superalgebra anti-automorphism of $\mathrm{X}(\mathfrak{p}_N)$. For instance, proving that a graded map $(-)^{\circ}\colon \mathrm{X}(\mathfrak{p}_N)\to\mathrm{X}(\mathfrak{p}_N)$ is a superalgebra anti-morphism is equivalent to showing the relation

$$R^{\mathsf{P}}(u,v)T_2^{\circ}(v)T_1^{\circ}(u)=T_1^{\circ}(u)T_2^{\circ}(v)R^{\mathsf{P}}(u,v),$$

where $T^{\circ}(u)=\sum_{i,j\in I_N}(-1)^{[i][j]+[j]}E_{ij}\otimes T_{ij}^{\circ}(u)$ and $T_k^{\circ}(u)$, $k=1,2$ are defined in the suitable ways. In our case, one can achieve this by modifying the $RTT$-relation (4.2.17) in suitable ways and using the stated properties of the $R$-matrix $R^{\mathsf{P}}(u,v)$.

### 4.2.3 The Hopf structure and central series $\mathcal{Z}(u)$ of $\mathrm{X}(\mathfrak{p}_N)$

The extended Yangian $\mathrm{X}(\mathfrak{p}_N)$ comes equipped with a Hopf superalgebra structure as given by the comultiplication

$$\Delta\colon\ \mathrm{X}(\mathfrak{p}_N)\to\mathrm{X}(\mathfrak{p}_N)\otimes\mathrm{X}(\mathfrak{p}_N),\quad T(u)\mapsto T_{[1]}(u)T_{[2]}(u),$$

the counit

$$\varepsilon\colon\ \mathrm{X}(\mathfrak{p}_N)\to\mathbb{C},\quad T(u)\mapsto\mathbb{1},$$

and the antipode

$$S\colon\ \mathrm{X}(\mathfrak{p}_N)\to\mathrm{X}(\mathfrak{p}_N),\quad T(u)\mapsto T(u)^{-1}.$$

Let us define $Z(u):=T^{\jmath^{\mathsf{P}}}(u)T(-u)$ and further consider the series $\mathcal{Z}(u)$ lying in $\mathrm{X}(\mathfrak{p}_N)[\![u^{-1}]\!]$ such that $\mathrm{id}\otimes\mathcal{Z}(u)=Z(u)$. Multiplying both sides of the $RTT$-relation by $-(u+v)$ and setting $v=-u$ yields the equation

$$QT_1(u)T_2(-u) = T_2(-u)T_1(u)Q. \tag{4.2.23}$$

Using that $QT_1(u) = QT_2^{j^{\mathsf{P}}}(u)$ and $T_1(u)Q = T_2^{j^{\mathsf{P}}}(u)Q$ and applying the map $(-)^{j^{\mathsf{P}}}$ to the first tensor factor of (4.2.23), we deduce

$$P \otimes \mathcal{Z}(u) = PT_2^{j^{\mathsf{P}}}(u)T_2(-u) = T_2(-u)T_2^{j^{\mathsf{P}}}(u)P.$$

Multiplying the above on the left by $P$, we obtain $\mathrm{id}^{\otimes 2} \otimes \mathcal{Z}(u) = T_2^{j^{\mathsf{P}}}(u)T_2(-u)$. Similarly, if we instead multiply the above equation on the right by $P$, we yield the relation $\mathrm{id}^{\otimes 2} \otimes \mathcal{Z}(u) = T_2(-u)T_2^{j^{\mathsf{P}}}(u)$. Therefore,

$$Z(u) = T^{j^{\mathsf{P}}}(u)T(-u) = T(-u)T^{j^{\mathsf{P}}}(u), \tag{4.2.24}$$

or rather put,

$$\delta_{ij}\mathcal{Z}(u) = \sum_{k \in I_N} T_{ik}^{j^{\mathsf{P}}}(u)T_{kj}(-u) = \sum_{k \in I_N} T_{ik}(-u)T_{kj}^{j^{\mathsf{P}}}(u), \tag{4.2.25}$$

where $\mathcal{Z}(u) = \mathbf{1} + \sum_{n=1}^{\infty} \mathcal{Z}_n u^{-n} \in \mathrm{X}(\mathfrak{p}_N)[\![u^{-1}]\!]$. We note that the coefficients of $\mathcal{Z}(u)$ are homogeneous of even degree, so all coefficients of $\mathcal{Z}(u)$ lies within the even subalgebra of $\mathrm{X}(\mathfrak{p}_N)$. Let us denote $\mathrm{ZX}(\mathfrak{p}_N)$ to be the subalgebra generated by the coefficients of $\mathcal{Z}(u)$ and let $(\mathcal{Z}(u) - \mathbf{1})$ to mean the two-sided graded ideal of $\mathrm{X}(\mathfrak{p}_N)$ generated by the coefficients of $\mathcal{Z}(u) - \mathbf{1}$. We now consider the following proposition which was first established in part by [Naz92].

**Proposition 4.2.5.** *The coefficients of the series* $\mathcal{Z}(u) \in \mathbf{1} + u^{-1}\mathrm{X}(\mathfrak{p}_N)[\![u^{-1}]\!]$ *given by the equation* $\mathrm{id} \otimes \mathcal{Z}(u) = T^{j^{\mathsf{P}}}(u)T(-u) = T(-u)T^{j^{\mathsf{P}}}(u)$ *lie in the center of* $\mathrm{X}(\mathfrak{p}_N)$. *Furthermore,*

$$\Delta \colon \mathcal{Z}(u) \mapsto \mathcal{Z}(u) \otimes \mathcal{Z}(u), \tag{4.2.26}$$

*where* $\Delta$ *is the comultiplication map on* $\mathrm{X}(\mathfrak{p}_N)$. *In particular,* $\mathrm{ZX}(\mathfrak{p}_N)$ *is a sub-Hopf superalgebra and* $(\mathcal{Z}(u) - \mathbf{1})$ *is a graded Hopf ideal of* $\mathrm{X}(\mathfrak{p}_N)$.

*Proof.* The proof is similar to [AAC+03, Theorem 3.1], but we shall provide it here.

First, we observe

$$(\mathrm{id}\otimes Z(u))T_2(v) = T_1^{j^{\mathsf{P}}}(u)T_1(-u)T_2(v) = T_1^{j^{\mathsf{P}}}(u)R^{\mathsf{P}}(-u,v)^{-1}T_2(v)T_1(-u)R^{\mathsf{P}}(-u,v),$$

by the $RTT$-relation. By applying $(-)^{j^{\mathsf{P}}}$ to the first tensor factor of the $RTT$-relation (4.2.17) and using the unitarity property (4.2.6), one yields the equation

$$T_1^{j^{\mathsf{P}}}(u)R^{\mathsf{P}}(-u,v)^{-1}T_2(v) = T_2(v)R^{\mathsf{P}}(-u,v)^{-1}T_1^{j^{\mathsf{P}}}(u).$$

Therefore,

$$\begin{aligned}(\mathrm{id}\otimes Z(u))T_2(v) &= T_2(v)R^{\mathsf{P}}(-u,v)^{-1}T_1^{j^{\mathsf{P}}}(u)T_1(-u)R^{\mathsf{P}}(-u,v)\\ &= T_2(v)R^{\mathsf{P}}(-u,v)^{-1}(\mathrm{id}\otimes Z(u))R^{\mathsf{P}}(-u,v) = T_2(v)(\mathrm{id}\otimes Z(u)),\end{aligned}$$

since $\mathrm{id}\otimes Z(u)$ commutes with $R^{\mathsf{P}}(-u,v)\otimes \mathbf{1}$. Furthermore, $\Delta\colon \mathcal{Z}(u)\mapsto \mathcal{Z}(u)\otimes\mathcal{Z}(u)$ is readily verified, since

$$\begin{aligned}\Delta(\mathcal{Z}(u)) &= \sum_{a,b,k\in I_N}(-1)^{[k][j]+[k]}\big(T_{ia}(-u)\otimes T_{ak}(-u)\big)\big(T_{-j,-b}(u)\otimes T_{-b,-k}(u)\big)\\ &= \sum_{a,b,k\in I_N}(-1)^{[k][j]+[k]+([a]+[k])([j]+[b])}T_{ia}(-u)T_{-j,-b}(u)\otimes T_{ak}(-u)T_{-b,-k}(u)\\ &= \sum_{a,b\in I_N}(-1)^{[a][j]+[a]}T_{ia}(-u)T_{-j,-a}(u)\otimes \delta_{ab}\mathcal{Z}(u) = \mathcal{Z}(u)\otimes\mathcal{Z}(u).\end{aligned}$$

Let us set $\mathcal{I} = (\mathcal{Z}(u)-\mathbf{1})$. One may verify that $\varepsilon\colon \mathcal{Z}(u)\mapsto 1$ and so $\varepsilon(\mathcal{I}) = 0$. Moreover, since $\Delta(\mathcal{Z}_n) = \sum_{a+b=n}\mathcal{Z}_a\otimes\mathcal{Z}_b$ (where $\mathcal{Z}_0 = \mathbf{1}$), then for $X\in \mathrm{X}(\mathfrak{p}_N)$ we have $\Delta(X\mathcal{Z}_n), \Delta(\mathcal{Z}_nX)\in\mathcal{I}\otimes\mathrm{X}(\mathfrak{p}_N)+\mathrm{X}(\mathfrak{p}_N)\otimes\mathcal{I}$, so $\mathcal{I}$ is a coideal. Lastly, the axioms of the Hopf superalgebra structure infer that the image of $\mathcal{Z}(u)$ under the antipode is given by

$$S\colon \mathcal{Z}(u)\mapsto\mathcal{Z}(u)^{-1},$$

which proves the proposition. □

By identifying $\mathcal{Z}(u)$ with $Z(u)$, equation (4.2.24) shows that the inverse of $T(u)$ is given by

$$T(u)^{-1} = \mathcal{Z}(-u)^{-1}T^{j^{\mathsf{P}}}(-u), \tag{4.2.27}$$

so the antipode on $\mathrm{X}(\mathfrak{p}_N)$ is the mapping $T(u)\mapsto\mathcal{Z}(-u)^{-1}T^{j^{P}}(-u)$. In particular, the square of the antipode is given by

$$S^2\colon T(u) \mapsto \mathcal{Z}(-u)\mathcal{Z}(u)^{-1}T(u). \tag{4.2.28}$$

### 4.2.4 The Yangian $\mathrm{Y}(\mathfrak{p}_N)$

We re-arrive at the definition of the Yangian for $\mathfrak{p}_N$:

**Definition 4.2.6.** The *Yangian* $\mathrm{Y}(\mathfrak{p}_N)$ of $\mathfrak{p}_N$ is the quotient of $\mathrm{X}(\mathfrak{p}_N)$ by the two-sided ideal $(\mathcal{Z}(u)-\mathbf{1})$, i.e.,

$$\mathrm{Y}(\mathfrak{p}_N) := \mathrm{X}(\mathfrak{p}_N)/(\mathcal{Z}(u)-\mathbf{1}).$$

For $i,j \in I_N$, $n \in \mathbb{Z}^+$, letting $\mathcal{T}_{ij}^{(n)}$ denote the image of the generator $T_{ij}^{(n)}$ under the canonical projection $\mathrm{X}(\mathfrak{p}_N) \twoheadrightarrow \mathrm{Y}(\mathfrak{p}_N)$ shows that the Yangian $\mathrm{Y}(\mathfrak{p}_N)$ coincides with Definition 4.2.1.

In terms of formal power series, the *RTT*-relation in Definition 4.2.1 is equivalent to the relations

$$\begin{aligned}\big[\mathcal{T}_{ij}(u), \mathcal{T}_{kl}(v)\big] = {} & \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\big(\mathcal{T}_{kj}(u)\mathcal{T}_{il}(v) - \mathcal{T}_{kj}(v)\mathcal{T}_{il}(u)\big) \\ & - \frac{1}{u+v}\Big(\delta_{i,-k}\sum_{p\in I_N}(-1)^{[i][j]+[j][p]+[p]}\mathcal{T}_{pj}(u)\mathcal{T}_{-p,l}(v) \\ & \qquad - \delta_{j,-l}\sum_{p\in I_N}(-1)^{[i][k]+[i]+[j][k]+[j]+[i][p]}\mathcal{T}_{k,-p}(v)\mathcal{T}_{ip}(u)\Big)\end{aligned} \tag{4.2.29}$$

for all $i,j,k,l \in I_N$ and relation (4.2.8) is equivalent to

$$\sum_{k\in I_N} \mathcal{T}_{ik}^{j\mathsf{P}}(u)\mathcal{T}_{kj}(-u) = \delta_{ij}\mathbf{1} \quad \text{for all} \quad i,j \in I_N. \tag{4.2.30}$$

Since $(\mathcal{Z}(u)-\mathbf{1})$ is a graded Hopf ideal, the quotient of $\mathrm{X}(\mathfrak{p}_N)$ by $(\mathcal{Z}(u)-\mathbf{1})$ comes equipped with a unique Hopf superalgebra structure such that the canonical projection $\mathrm{X}(\mathfrak{p}_N) \twoheadrightarrow \mathrm{X}(\mathfrak{p}_N)/(\mathcal{Z}(u)-\mathbf{1})$ is a Hopf superalgebra morphism. Hence, there is a Hopf superalgebra structure on $\mathrm{Y}(\mathfrak{p}_N)$ is given by the comultiplication

$$\Delta\colon\ \mathrm{Y}(\mathfrak{p}_N) \to \mathrm{Y}(\mathfrak{p}_N) \otimes \mathrm{Y}(\mathfrak{p}_N), \quad \mathcal{T}(u) \mapsto \mathcal{T}_{[1]}(u)\mathcal{T}_{[2]}(u),$$

the counit

$$\varepsilon\colon\ \mathrm{Y}(\mathfrak{p}_N)\to\mathbb{C},\quad \mathcal{T}(u)\mapsto\mathbb{1},$$

and the antipode

$$S\colon\ \mathrm{Y}(\mathfrak{p}_N)\to\mathrm{Y}(\mathfrak{p}_N),\quad \mathcal{T}(u)\mapsto\mathcal{T}(u)^{-1}=\mathcal{T}^{\jmath^{\mathsf{P}}}(-u).$$

We shall consider two ascending algebra filtrations on $\mathrm{Y}(\mathfrak{p}_N)$, which will be denoted $\mathbf{F}\big(\mathrm{Y}(\mathfrak{p}_N)\big)=\mathbf{F}=\{\mathbf{F}_n\}_{n\in\mathbb{N}}$ and $\mathbf{F}'\big(\mathrm{Y}(\mathfrak{p}_N)\big)=\mathbf{F}'=\{\mathbf{F}'_n\}_{n\in\mathbb{N}}$, given via the respective filtration degree assignments

$$\deg_{\mathbf{F}}\mathcal{T}_{ij}^{(n)}=n-1\quad\text{and}\quad\deg_{\mathbf{F}'}\mathcal{T}_{ij}^{(n)}=n. \tag{4.2.31}$$

for all $i,j\in I_N$ and $n\in\mathbb{Z}^+$. From the relations (4.2.29), one can deduce that the associated graded superalgebra $\mathrm{gr}_{\mathbf{F}'}\mathrm{Y}(\mathfrak{p}_N)=\bigoplus_{n\in\mathbb{N}}\mathbf{F}'_n/\mathbf{F}'_{n-1}$ is supercommutative. We shall direct our attention to the first filtration $\mathbf{F}$ which will induce a more relevant associated graded superalgebra:

$$\mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)=\mathrm{gr}_{\mathbf{F}}\mathrm{Y}(\mathfrak{p}_N)=\bigoplus_{n\in\mathbb{N}}\mathbf{F}_n/\mathbf{F}_{n-1},$$

We note that $\mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)$ inherits a $\mathbb{Z}_2$-graded structure from $\mathrm{Y}(\mathfrak{p}_N)$ by assigning $\mathbb{Z}_2$-grade $[i]+[j]$ to the image $\overline{\mathcal{T}}_{ij}^{(n)}$ of $\mathcal{T}_{ij}^{(n)}$ in $\mathbf{F}_{n-1}/\mathbf{F}_{n-2}$. Furthermore, $\mathbf{F}$ is a Hopf filtration, so $\mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)$ is endowed with an $\mathbb{N}$-graded Hopf superstructure given by the comultiplication

$$\begin{gathered}\mathrm{gr}\,\Delta\colon\ \mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)\to\mathrm{gr}\left(\mathrm{Y}(\mathfrak{p}_N)^{\otimes 2}\right)\cong\left(\mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)\right)^{\otimes 2}\\ \overline{\mathcal{T}}_{ij}^{(n)}\mapsto\overline{\mathcal{T}}_{ij}^{(n)}\otimes\mathbf{1}+\mathbf{1}\otimes\overline{\mathcal{T}}_{ij}^{(n)},\end{gathered}$$

the counit

$$\mathrm{gr}\,\varepsilon\colon\ \mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)\to\mathbb{C},\quad \overline{\mathcal{T}}_{ij}^{(n)}\mapsto 0,$$

and antipode

$$\mathrm{gr}\,S\colon\ \mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)\to\mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N),\quad \overline{\mathcal{T}}_{ij}^{(n)}\mapsto-\overline{\mathcal{T}}_{ij}^{(n)}.$$

for all $i,j\in I_N$ and $n\in\mathbb{Z}^+$.

Letting $\mathfrak{gl}_{N|N}[z]=\mathfrak{gl}_{N|N}\otimes\mathbb{C}[z]$ denote the polynomial current Lie superalgebra associated to $\mathfrak{gl}_{N|N}$, the involution $(-)^{\iota^{\mathsf{P}}}$ may be extended to an involutory automorphism on $\mathfrak{gl}_{N|N}[z]$, also denoted $(-)^{\iota^{\mathsf{P}}}$, by assigning

$$(X \otimes f(z))^{\iota^{\mathsf{P}}} = X^{\iota^{\mathsf{P}}} \otimes f(-z) \quad \text{for all} \quad X \in \mathfrak{gl}_{N|N}, \quad f(z) \in \mathbb{C}[z].$$

The *twisted current Lie superalgebra* $\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}$ is defined as the fixed-point Lie sub-superalgebra of $\mathfrak{gl}_{N|N}[z]$ under the involutive automorphism $(-)^{\iota^{\mathsf{P}}}$:

$$\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}} := \{g(z) \in \mathfrak{gl}_{N|N}[z] \mid g(z)^{\iota^{\mathsf{P}}} = g(z)\}.$$

Using the identification $Xz^n = X \otimes z^n$ for elements in $\mathfrak{gl}_{N|N}[z]$, we find that $\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}$ is generated by the operators

$$\mathsf{E}_{ij}^{(n)}(z) := E_{ij}z^n + E_{ij}^{\iota^{\mathsf{P}}}(-z)^n = (E_{ij} - (-1)^{[i][j]+[i]+n}E_{-j,-i})z^n \in \mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}} \tag{4.2.32}$$

with $i, j \in I_N$, $n \in \mathbb{N}$, subject only to the relations

$$\begin{aligned}\left[\mathsf{E}_{ij}^{(m)}(z), \mathsf{E}_{kl}^{(n)}(z)\right] &= \delta_{jk}\mathsf{E}_{il}^{(m+n)}(z) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}\mathsf{E}_{kj}^{(m+n)}(z) \\ &\quad - \delta_{i,-k}(-1)^{[i][j]+[i]+m}\mathsf{E}_{-j,l}^{(m+n)}(z) + \delta_{j,-l}(-1)^{([i]+[j])[k]+m}\mathsf{E}_{k,-i}^{(m+n)}(z)\end{aligned} \tag{4.2.33}$$

and

$$\mathsf{E}_{ij}^{(n)}(z) + (-1)^{[i][j]+[i]+n}\mathsf{E}_{-j,-i}^{(n)}(z) = 0. \tag{4.2.34}$$

We now have the following proposition:

**Proposition 4.2.7.** *There is an $\mathbb{N}$-graded Hopf superalgebra epimorphism*

$$\Phi \colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \to \operatorname{gr} \mathrm{Y}(\mathfrak{p}_N), \quad \mathsf{E}_{ij}^{(n-1)}(z) \mapsto -(-1)^{[i][j]}\overline{\mathcal{T}}_{ji}^{(n)} \tag{4.2.35}$$

*for all $i, j \in I_N$ and $n \in \mathbb{Z}^+$.*

*Proof.* To show $\Phi \colon \mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}} \to \operatorname{Lie}(\operatorname{gr} \mathrm{Y}(\mathfrak{p}_N))$ is an $\mathbb{N}$-graded Lie superalgebra morphism, one passes the defining relations (4.2.29) and (4.2.30) to the associated graded superalgebra and uses the expansions

$$\frac{1}{u \mp v} = \frac{u^{-1}}{1 \mp u^{-1}v} = u^{-1}\sum_{r=0}^{\infty}(\pm u^{-1}v)^r$$

to yield the relations

$$\begin{aligned}\left[\overline{\mathcal{T}}_{ji}^{(m)}, \overline{\mathcal{T}}_{lk}^{(n)}\right] &= -\delta_{jk}(-1)^{[i][j]+[i][l]+[k][l]}\overline{\mathcal{T}}_{li}^{(m+n-1)} + \delta_{il}(-1)^{[i]}\overline{\mathcal{T}}_{jk}^{(m+n-1)} \\ &\quad + \delta_{i,-k}(-1)^{[i][l]+[i]+[j][l]+m-1}\overline{\mathcal{T}}_{l,-j}^{(m+n-1)} - \delta_{j,-l}(-1)^{[i][j]+m-1}\overline{\mathcal{T}}_{-i,k}^{(m+n-1)}.\end{aligned}$$

and

$$(-1)^n\overline{\mathcal{T}}_{ji}^{(n)} + (-1)^{[i][j]+[j]}\overline{\mathcal{T}}_{-i,-j}^{(n)} = 0$$

for all $i, j \in I_N$ and $m, n \in \mathbb{Z}^+$. The desired relations follow from multiplying the first relation by the scalar $(-1)^{[i][j]+[k][l]}$ and the second by $-(-1)^{[i][j]+n}$.

Hence, $\Psi$ extends to a superalgebra morphism $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \to \operatorname{gr} \mathrm{Y}(\mathfrak{p}_N)$, which is also $\mathbb{N}$-graded. Such morphism is surjective since $\operatorname{gr} \mathrm{Y}(\mathfrak{p}_N)$ is generated by the elements $\overline{\mathcal{T}}_{ij}^{(n)}$. Lastly, it can be seen that $\Psi$ is a morphism of Hopf superalgebras from the descriptions of those Hopf superstructures on $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ and $\operatorname{gr} \mathrm{Y}(\mathfrak{p}_N)$ as before. □

## 4.3 Poincaré-Birkhoff-Witt Theorem for Yangians of Type $P$

In this section, we illustrate how to obtain an explicit algebraic basis for the Yangian $\mathrm{Y}(\mathfrak{p}_N)$ which amounts to proving the Yangian is a filtered deformation of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$. Indeed, suppose such an isomorphism $\Phi\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \xrightarrow{\sim} \operatorname{gr} \mathrm{Y}(\mathfrak{p}_N)$ exists, where $\operatorname{gr} \mathrm{Y}(\mathfrak{p}_N)$ is the associated graded superalgebra induced by the filtration $\mathbf{F}$ as described by (4.2.31). The Poincaré-Birkhoff-Witt Theorem for Lie superalgebras infers one can construct a basis $\mathbf{B}$ for $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$, so any lift of $\Phi(\mathbf{B})$ will yield the desired basis for the Yangian.

### 4.3.1 Evaluation and $R$-matrix representations

Given the vector representation $\rho\colon \mathfrak{U}(\mathfrak{gl}_{N|N}) \to \operatorname{End} \mathbb{C}^{N|N}$, one can pullback $\rho$ by the superalgebra morphism $\mathrm{ev}_a\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]) \to \mathfrak{U}(\mathfrak{gl}_{N|N})$ induced by the assignment $z \mapsto a$ to yield the *evaluation representation* of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$ at $a \in \mathbb{C}$ given by

$$\rho_a := \mathrm{ev}_a^*\,\rho\colon \mathfrak{U}(\mathfrak{gl}_{N|N}) \to \operatorname{End} \mathbb{C}^{N|N}, \quad E_{ij} \mapsto a^n \rho(E_{ij})$$

for all $i, j \in I_N$. For any complex numbers $a_1, \ldots, a_n \in \mathbb{C}$, we consider the tensor product of $2n$ evaluation representations of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$ as described by

$$\rho_{a_1 \to a_n} := (\otimes_{i=1}^{n} (\rho_{a_i} \otimes \rho_{-a_i})) \circ \Delta_{2n-1}, \tag{4.3.1}$$

where $\Delta_{2n-1} \colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]) \to \mathfrak{U}(\mathfrak{gl}_{N|N}[z])^{\otimes 2n}$ is the unique $(2n-1)$-fold coproduct sending $X \in \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ to the element $\sum_{(X)} X_{(1)} \otimes X_{(2)} \otimes \cdots \otimes X_{(2n)}$ in Sweedler notation.

The following lemma establishes that the intersection of the kernels of all such representations $\rho_{a_1 \to a_n}$, $a_1, \ldots, a_n \in \mathbb{C}$ is trivial. The core ideas for the proof come from the proofs of similar statements in [Naz99, Proposition 2.2] and [AMR06, Lemma 3.5].

**Lemma 4.3.1.** $\bigcap_{n \in \mathbb{Z}^+} \bigcap_{(a_1, \ldots, a_n) \in \mathbb{C}^n} \ker(\rho_{a_1 \to a_n}) = 0$ *in* $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$.

*Proof.* Let $\{X_i\}_{i=1}^{4N^2}$ be an ordered homogeneous basis of $\mathfrak{gl}_{N|N}$ such that $X_1 = \mathrm{id}$, where we will write $\chi_i = \rho(X_i)$ for all indices $i = 1, 2, \ldots, 4N^2$. Furthermore, we shall let $\{\mathfrak{U}_n(\mathfrak{gl}_{N|N}[z])\}_{n \in \mathbb{N}}$ denote the canonical ascending algebra filtration on $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$ determined by monomial length.

*Step 1.* Given the ordering on $\{X_i\}_{i=1}^{4N^2}$, there will be an induced total ordering '$\preceq$' on the basis $\{X_b z^m \mid 1 \leq b \leq 4N^2,\ m \in \mathbb{N}\}$ of $\mathfrak{gl}_{N|N}[z]$. The Poincaré-Birkhoff-Witt Theorem for Lie superalgebras states that its universal enveloping superalgebra $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$ therefore has a basis consisting of ordered monomials of the form $\prod_{j=1}^{r} X_{b_j} z^{m_j}$ such that $X_{b_j} z^{m_j} \preceq X_{b_{j+1}} z^{m_{j+1}}$ for indices $j = 1, \ldots, r-1$, and $X_{b_j} z^{m_j} \neq X_{b_{j+1}} z^{m_{j+1}}$ if $[X_{b_j}] = \bar{1}$. Given a nonzero element $A$ in $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$, we may therefore express such element as a unique linear combination of PBW basis monomials in $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$ and we shall let $\{M_i = \prod_{j=1}^{n} X_{b_{ij}} z^{m_{ij}}\}_{i=1}^{p}$ denote the collection of those basis elements with maximal filtration degree $n$.

For each index $1 \leq k \leq n$, we shall set $\widehat{z}_{2k-1} = z_k$ and $\widehat{z}_{2k} = -z_k$ where $z_1, \ldots, z_n$ are formal variables. In particular, any $n$ integers $1 \leq r_1 < \cdots < r_n \leq 2n$ determine an embedding

$$\begin{aligned} \nu_{r_1, \ldots, r_n} \colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z])^{\otimes n} &\to \bigotimes_{k=1}^{n} \mathfrak{U}(\mathfrak{gl}_{N|N}[z_k]) \otimes \mathfrak{U}(\mathfrak{gl}_{N|N}[-z_k]) \\ Y_1(z) \otimes \cdots \otimes Y_n(z) &\mapsto 1^{\otimes(r_1-1)} \otimes Y_1(\widehat{z}_{r_1}) \otimes 1^{\otimes(r_2-r_1-1)} \otimes \cdots \otimes Y_n(\widehat{z}_{r_n}) \otimes 1^{\otimes(2n-r_n)}, \end{aligned}$$

where $Y_k(z)$, $1 \leq k \leq n$, are monomials in $\mathfrak{U}(\mathfrak{gl}_{N|N}[z])$.

Thus, for each monomial $M_i$, $1 \leq i \leq p$, we associate the supersymmetrized object

$$M_i^\sigma := \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma, M_i)} \sum_{1 \leq r_1 < \cdots < r_n \leq 2n} \nu_{r_1,\ldots,r_n}\big(\otimes_{j=1}^n X_{b_{i\sigma(j)}} z^{m_{i\sigma(j)}}\big), \tag{4.3.2}$$

where $(-1)^{\epsilon(\sigma, M_i)}$ is the Koszul sign provided that $\epsilon \colon \mathfrak{S}_n \times (\mathfrak{gl}_{N|N}[z])^{\otimes n} \to \mathbb{Z}_2$ is the map defined by the assignment $\epsilon(\sigma, x) = \sum_{(k,l) \in \mathrm{Inv}(\sigma)} [x_{\sigma(k)}][x_{\sigma(l)}]$ for homogeneous tensors $x = x_1 \otimes \cdots \otimes x_n \in (\mathfrak{gl}_{N|N}[z])^{\otimes n}$ and where $\mathrm{Inv}(\sigma) = \{(k,l) \mid k < l,\ \sigma(k) > \sigma(l)\}$ is the set of inversions.

*Step 2.* For each current Lie superalgebra $\mathfrak{gl}_{N|N}[\widehat{z}_k]$, $1 \leq k \leq 2n$, we endow a total ordering on its basis $\{X_b \widehat{z}_k^m \mid 1 \leq b \leq 4N^2,\ m \in \mathbb{N}\}$ in a similar way to before so that we obtain a basis $\mathbf{B}$ of $\bigotimes_{k=1}^{2n} \mathfrak{U}(\mathfrak{gl}_{N|N}[\widehat{z}_k])$ consisting of elements of the form $\otimes_{k=1}^{2n} X_{b_{k,1}} \widehat{z}_i^{m_{k,1}} \cdots X_{b_{k,h_k}} \widehat{z}_i^{m_{k,h_k}}$, where $X_{b_{k,1}} \widehat{z}_i^{m_{k,1}} \cdots X_{b_{k,h_k}} \widehat{z}_i^{m_{k,h_k}}$ is a PBW basis monomial for $\mathfrak{gl}_{N|N}[\widehat{z}_k]$. Considering now the linear map

$$\begin{gathered}
\phi \colon \bigotimes_{k=1}^{2n} \mathfrak{U}(\mathfrak{gl}_{N|N}[\widehat{z}_k]) \to \mathfrak{U}(\mathfrak{gl}_{N|N})^{\otimes 2n}[z_1, \ldots, z_n] \\
\otimes_{k=1}^{2n} X_{b_{k,1}} \widehat{z}_i^{m_{k,1}} \cdots X_{b_{k,h_k}} \widehat{z}_i^{m_{k,h_k}} \mapsto \big(\otimes_{i=1}^{2n} X_{b_{k,1}} \cdots X_{b_{k,h_k}}\big) \prod_{k=1}^{2n} \widehat{z}_k^{m_{k,1}} \cdots \widehat{z}_k^{m_{k,h_k}},
\end{gathered}$$

we claim that the elements $\phi(M_i^\sigma)$, $i = 1, \ldots, p$, are linearly independent. Noting that each term in the sum $\phi(M_i^\sigma)$ is an element of the basis $\mathbf{B}$ up to sign, it suffices to show that there exists a basis element (up to scaling) in each expression $\phi(M_i^\sigma)$ that does not occur in any other expressions $\phi(M_k^\sigma)$ for $k \neq i$. In fact, we observe that such a candidate is

$$\phi\big(\nu_{1,3,\ldots,2n-1}\big(\otimes_{j=1}^n X_{b_{ij}} z^{m_{ij}}\big)\big) = (X_{b_{i1}} \otimes 1 \otimes \cdots \otimes X_{b_{in}} \otimes 1) z_1^{m_{i1}} \cdots z_n^{m_{in}}, \tag{4.3.3}$$

since the elements $\widetilde{M_i^\sigma} := \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma, M_i)} \otimes_{j=1}^n X_{b_{i\sigma(j)}} z^{m_{i\sigma(j)}}$, $i = 1, \ldots, p$, are linearly independent.

*Step 3.* For any complex numbers $a_1, \ldots, a_n \in \mathbb{C}$ and each index $1 \leq k \leq n$, we shall set $\widehat{a}_{2k-1} = a_k$ and $\widehat{a}_{2k} = -a_k$. In particular, as

$$\rho_{a_1 \to a_n}(X_b z^m) = \sum_{k=1}^{2n} \widehat{a}_k^m \chi_b^{[k]}, \qquad \chi_b^{[k]} := \mathrm{id}^{\otimes(k-1)} \otimes \chi_b \otimes \mathrm{id}^{\otimes(n-k)} \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n},$$

then the image of any monomial $\prod_{j=1}^{r} X_{b_j} z^{m_j}$ under $\rho_{a_1 \to a_n}$ will be given by

$$\sum_{k_1,\dots,k_r=1}^{2n} \widehat{a}_{k_1}^{m_1} \cdots \widehat{a}_{k_r}^{m_r} \chi_{b_1}^{[k_1]} \cdots \chi_{b_r}^{[k_r]} \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}. \tag{4.3.4}$$

Consider now the subspace of $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}$ given by

$$W_{2n} := \mathrm{span}_{\mathbb{C}} \left\{ \chi_{i_1} \otimes \cdots \otimes \chi_{i_{2n}} \mid \chi_1 = \mathrm{id} \text{ occurs in at least } n{+}1 \text{ tensor factors} \right\},$$

where $1 \leq i_k \leq 4N^2$ for $1 \leq k \leq 2n$. We observe that the image of any element in $\mathfrak{U}_{n-1}(\mathfrak{gl}_{N|N}[z])$ under $\rho_{a_1 \to a_n}$ will be contained in the subspace $W_{2n}$. Moreover, since any $n$ integers $1 \leq r_1 < \cdots < r_n \leq 2n$ determines an embedding

$$\begin{aligned} \upsilon_{r_1,\dots,r_n} \colon \ & \mathrm{End}(\mathbb{C}^{N|N})^{\otimes n} \to \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n} \\ & \varphi_1 \otimes \cdots \otimes \varphi_n \mapsto \mathrm{id}^{\otimes (r_1 - 1)} \otimes \varphi_1 \otimes \mathrm{id}^{\otimes (r_2 - r_1 - 1)} \otimes \cdots \otimes \varphi_n \otimes \mathrm{id}^{\otimes (2n - r_n)}, \end{aligned}$$

we can use (4.3.4) to express the image of the monomial $M_i$ under $\rho_{a_1 \to a_n}$ as

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma, M_i)} \sum_{1 \leq r_1 < \cdots < r_n \leq 2n} \widehat{a}_{r_1}^{m_{i\sigma(1)}} \cdots \widehat{a}_{r_n}^{m_{i\sigma(n)}} \upsilon_{r_1,\dots,r_n} \Big( \bigotimes_{j=1}^{n} \chi_{b_{i\sigma(j)}} \Big) \mod W_{2n}. \tag{4.3.5}$$

Since $\rho$ is a faithful representation, then so is $\rho^{\otimes 2n} \colon \mathfrak{U}(\mathfrak{gl}_{N|N})^{\otimes 2n} \to \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}$ and its extension to $\mathfrak{U}(\mathfrak{gl}_{N|N})^{\otimes 2n}[z_1, \dots, z_n] \to \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}[z_1, \dots, z_n]$, which we also denote $\rho^{\otimes 2n}$. Therefore, since the elements $\phi(M_i^{\sigma})$, $i = 1, \dots, p$, are linearly independent, then their images under $\rho^{\otimes n}$ are so. That is, a nonzero linear combination $\sum_{i=1}^{p} \lambda_i \phi(M_i^{\sigma})$ implies that the sum of polynomials

$$\sum_{i=1}^{p} \lambda_i \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma, M_i)} \sum_{1 \leq r_1 < \cdots < r_n \leq 2n} \upsilon_{r_1,\dots,r_n} \Big( \bigotimes_{j=1}^{n} \chi_{b_{i\sigma(j)}} \Big) \widehat{z}_{r_1}^{m_{i\sigma(1)}} \cdots \widehat{z}_{r_n}^{m_{i\sigma(n)}}$$

is nonzero. Hence, there exists complex numbers $a_1, \dots, a_n \in \mathbb{C}$ such that

$$\sum_{i=1}^{p} \lambda_i \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma, M_i)} \sum_{1 \leq r_1 < \cdots < r_n \leq 2n} \widehat{a}_{r_1}^{m_{i\sigma(1)}} \cdots \widehat{a}_{r_n}^{m_{i\sigma(n)}} \upsilon_{r_1,\dots,r_n} \Big( \bigotimes_{j=1}^{n} \chi_{b_{i\sigma(j)}} \Big)$$

is nonzero. Comparing the above with (4.3.5), we conclude that that image of $\rho_{a_1 \to a_n}(A)$ in the quotient $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n} / W_{2n}$ is nonzero and therefore $\rho_{a_1 \to a_n}(A) \neq 0$, proving the lemma. □

For any $a \in \mathbb{C}$, restricting the evaluation representation $\rho_a$ to $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ via the inclusion $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \hookrightarrow \mathfrak{U}(\mathfrak{gl}_{N|N}[z])$ will give rise to a corresponding evaluation representation which we also denote by $\rho_a$:

$$\rho_a \colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \to \operatorname{End} \mathbb{C}^{N|N}, \quad \mathsf{E}_{ij}^{(n)}(z) \mapsto a^n E_{ij} + (-a)^n E_{ij}^{\iota^{\mathsf{P}}} \tag{4.3.6}$$

for all $i, j \in I_N$ and $n \in \mathbb{N}$. Accordingly, by regarding $\rho_{a_1 \to a_n}$ for $a_1, \ldots, a_n \in \mathbb{C}$ as a representation of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ by restriction, it follows from the previous lemma that we have $\bigcap_{n \in \mathbb{Z}^+} \bigcap_{(a_1,\ldots,a_n) \in \mathbb{C}^n} \ker(\rho_{a_1 \to a_n}) = 0$ in $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ as well.

We will now direct our attention to a canonical representation of the extended Yangian $\mathrm{X}(\mathfrak{p}_N)$ called the *R-matrix representation*. This representation will give rise to an important representation of the Yangian $\mathrm{Y}(\mathfrak{p}_N)$ which will be used to prove the isomorphism $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \cong \operatorname{gr} \mathrm{Y}(\mathfrak{p}_N)$. For any $a \in \mathbb{C}$, such $R$-matrix representation at $a \in \mathbb{C}$ is given by

$$R_a \colon \mathrm{X}(\mathfrak{p}_N) \to \operatorname{End} \mathbb{C}^{N|N}, \quad T(u) \mapsto R(u, a). \tag{4.3.7}$$

In terms of formal power series, the $R$-matrix representation takes the form

$$R_a \colon T_{ji}(u) \mapsto -(-1)^{[i][j]} \left( -(-1)^{[i][j]} \mathbf{1} + \frac{E_{ij}}{u-a} - \frac{(-1)^{[i][j]+[i]} E_{-j,-i}}{u+a} \right),$$

for $i, j \in I_N$; hence, $R_a(T_{ji}^{(n)}) = -(-1)^{[i][j]} \rho_a(\mathsf{E}_{ij}^{(n-1)}(z))$ for $n \in \mathbb{Z}^+$. By the Hopf superalgebra structure on $\mathrm{X}(\mathfrak{p}_N)$, we may consider the tensor product of these representations: $(R_a \otimes R_{-a}) \circ \Delta \colon T(u) \mapsto R_{12}(u, a) R_{13}(u, -a)$. Considering the series $f_a(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ given by

$$f_a(u) = \frac{(u+a)^2}{(u+a)^2 - 1},$$

the pullback of the representation $(R_a \otimes R_{-a}) \circ \Delta$ by the shift automorphism $\mu_{f_a}$ (4.2.19) yields a new representation $\phi_a := \mu_{f_a}^*((R_a \otimes R_{-a}) \circ \Delta)$ of $\mathrm{X}(\mathfrak{p}_N)$ given by

$$\phi_a \colon \mathrm{X}(\mathfrak{p}_N) \to \operatorname{End}(\mathbb{C}^{N|N})^{\otimes 2}, \quad T(u) \mapsto f_a(u) R_{12}(u, a) R_{13}(u, -a).$$

In particular, we find that $\phi_a(Z(u)) = \mathrm{id}^{\otimes 2}$. Indeed, by the unitarity property (4.2.6) of the $R$-matrix, we have

$$\begin{aligned}\phi_a(T^{j^{\mathsf{P}}}(u)T(-u)) &= f_a(u)f_a(-u)R_{1,3}(u,-a)^{j_1^{\mathsf{P}}}R_{12}(u,a)^{j_1^{\mathsf{P}}}R_{12}(-u,a)R_{13}(-u,-a)\\ &= f_a(u)f_a(-u)f_a(u)^{-1}f_a(-u)^{-1}\,\mathrm{id}^{\otimes 2n} = \mathrm{id}^{\otimes 2n}.\end{aligned}$$

Therefore, the representation $\phi_a$ descends to a representation of the Yangian:

$$\varphi_a\colon \mathrm{Y}(\mathfrak{p}_N) \to \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2}, \quad \mathcal{T}(u) \mapsto f_a(u)R_{12}(u,a)R_{13}(u,-a). \tag{4.3.8}$$

### 4.3.2 The PBW Theorem and supercenter of $\mathrm{Y}(\mathfrak{p}_N)$

We are now in position to prove that $\mathrm{Y}(\mathfrak{p}_N)$ is a filtered deformation of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$. The proof of the following theorem is similar to [AMR06, Theorem 3.6], which leverages the lemma introduced in the previous subsection.

**Theorem 4.3.2.** *The epimorphism in Proposition 4.2.7 is an $\mathbb{N}$-graded Hopf superalgebra isomorphism*

$$\Phi\colon \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \xrightarrow{\sim} \mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N), \quad \mathsf{E}_{ij}^{(n-1)}(z) \mapsto -(-1)^{[i][j]}\mathcal{T}_{ji}^{(n)} \tag{4.3.9}$$

*for $i,j \in I_N$ and $n \in \mathbb{Z}^+$.*

*Proof.* By Proposition (4.2.7), all that is left to show is injectivity. To this end, we let $A \in \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ be a nonzero homogeneous element of gradation degree $d$; that is,

$$A = \sum A^{k_1;\dots;k_m}_{i_1j_1;\dots;i_mj_m}\mathsf{E}^{(k_1-1)}_{i_1j_1}(z)\cdots\mathsf{E}^{(k_m-1)}_{i_mj_m}(z), \quad \text{where} \quad A^{k_1;\dots;k_m}_{i_1j_1;\dots;i_mj_m} \in \mathbb{C},$$

and the summation indices $i_b, j_b, k_b$, $1 \le b \le m$, satisfy $i_b, j_b \in I_N$ and $\sum_{b=1}^m k_b = d+m$. Considering the element

$$\widetilde{A} = \sum (-1)^{m+\sum_{b=1}^m [i_b][j_b]} A^{k_1;\dots;k_m}_{i_1j_1;\dots;i_mj_m}\mathcal{T}^{(k_1)}_{j_1i_1}\cdots\mathcal{T}^{(k_m)}_{j_mi_m} \in \mathrm{Y}(\mathfrak{p}_N)$$

whose summation indices $i_b, j_b, k_b$, $1 \le b \le m$, satisfy the same conditions as above, then $\Phi(A)$ coincides with the image of $\widetilde{A}$ in $\mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)$, so it suffices to prove that the filtration degree of $\widetilde{A}$ is $d$.

*Step 1.* Via the expansion $f_a(u) = (u+a)^2/((u+a)^2-1) = \sum_{p=0}^{\infty}(u+a)^{-2p}$, where

$$\frac{1}{(u+a)^{2p}} = u^{-2p}\left(\sum_{n=0}^{\infty}(-a)^n u^{-n}\right)^{2p} = \sum_{n=2p}^{\infty}\binom{n-1}{n-2p}(-a)^{n-2p}u^{-n},$$

we see that the coefficient of $u^{-n}$ in $f_a(u)$ is given by $\sum_{p=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-1}{n-2p}(-a)^{n-2p}$. In particular, by regarding $a$ as a formal variable in $\mathbb{C}$, the coefficients of $f_a(u)$ are polynomials in $\mathbb{C}[a]$ with polynomial degrees given by $\deg_a f_a^{(1)} = 0$ and $\deg_a f_a^{(n)} = n-2$ for $n \geq 2$. Recalling the map $\rho_a$ (4.3.6), the image of $\mathcal{T}_{ij}^{(n)}$ under the representation (4.3.8) is given by

$$\begin{aligned}\varphi_a(\mathcal{T}_{ji}^{(n)}) = \delta_{ij} f_a^{(n)}\,\mathrm{id}^{\otimes 2} - (-1)^{[i][j]}\big(\rho_a(\mathsf{E}_{ij}^{(n-1)}(z)) \otimes \mathrm{id} + \mathrm{id}\otimes\rho_{-a}(\mathsf{E}_{ij}^{(n-1)}(z))\big)\\ + \sum_{r+s+t=n} f_a^{(r)}\rho_a(\mathsf{E}_{ij}^{(s-1)}(z)) \otimes \rho_{-a}(\mathsf{E}_{ij}^{(t-1)}(z)),\end{aligned}$$

where $r \in \mathbb{N}$ and $s,t,n \in \mathbb{Z}^+$. Thus, $\varphi_a(\mathcal{T}_{ji}^{(n)}) \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2}[a]$ with polynomial degree $n-1$, where its highest degree term is $-(-1)^{[i][j]}(\rho_a(\mathsf{E}_{ij}^{(n-1)}(z)) \otimes \mathrm{id} + \mathrm{id}\otimes\rho_a(\mathsf{E}_{ij}^{(n-1)}(z)))$.

*Step 2.* Given complex numbers $x_1,\ldots,x_n \in \mathbb{C}$, we consider the tensor product $\varphi_{x_1\to x_n} := (\bigotimes_{i=1}^n \varphi_{x_i}) \circ \Delta_{n-1}$. Equipping $\mathrm{Y}(\mathfrak{p}_N)^{\otimes n}$ with the tensor product filtration $\mathbf{F}^n = \{\mathbf{F}_h^n\}_{h\in\mathbb{N}}$ induced by the one on $\mathrm{Y}(\mathfrak{p}_N)$, i.e., $\mathbf{F}_h^n = \bigoplus_{\sum_{i=1}^n k_i = h}\mathbf{F}_{k_1}\otimes\cdots\otimes\mathbf{F}_{k_n}$, then writing the sum $\sum_{b=1}^m k_b = d+m$ allows one to express $\Delta_{n-1}(\mathcal{T}_{j_1i_1}^{(k_1)}\cdots\mathcal{T}_{j_mi_m}^{(k_m)})$ as

$$\sum\nolimits_{q_1,\ldots,q_m=1}^{n}(\mathcal{T}_{j_1i_1}^{(k_1)})_{[q_1]}\cdots(\mathcal{T}_{j_mi_m}^{(k_m)})_{[q_m]} \mod \mathbf{F}_{d-1}^n,$$

where $(\mathcal{T}_{i_bj_b}^{(k_b)})_{[q_b]} = \mathbf{1}^{\otimes(q_b-1)}\otimes\mathcal{T}_{i_bj_b}^{(k_b)}\otimes\mathbf{1}^{\otimes(n-q_b)}$ for $1\leq b\leq m$. Regarding $x_1,\ldots,x_n$ as formal variables taking values in $\mathbb{C}$, the image of the monomial $\mathcal{T}_{j_1i_1}^{(k_1)}\cdots\mathcal{T}_{j_mi_m}^{(k_m)}$ under the representation $\varphi_{x_1\to x_n}$ will lie in $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}[x_1,\ldots,x_n]$ with polynomial degree satisfying $\deg\big(\varphi_{x_1\to x_n}(\mathcal{T}_{j_1i_1}^{(k_1)}\cdots\mathcal{T}_{j_mi_m}^{(k_m)})\big) \leq d$. If $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}[x_1,\ldots,x_n]_{d-1}$ denotes the subspace of polynomials in $x_1,\ldots,x_n$ with degree at most $d-1$, the element $\varphi_{x_1\to x_n}(\mathcal{T}_{j_1i_1}^{(k_1)}\cdots\mathcal{T}_{j_mi_m}^{(k_m)})$ is equivalent modulo $\mathrm{End}(\mathbb{C}^{M|N})^{\otimes 2n}[x_1,\ldots,x_n]_{d-1}$ to the expression

$$\sum_{q_1,\ldots,q_m=1}^{n}(-1)^{m+\sum_{b=1}^m[i_b][j_b]}\prod_{b=1}^{m}\big(\rho_{x_{q_b}}(\mathsf{E}_{i_bj_b}^{(k_b-1)}(z))\otimes\mathrm{id}+\mathrm{id}\otimes\rho_{-x_{q_b}}(\mathsf{E}_{i_bj_b}^{(k_b-1)}(z))\big)^{[q_b]},$$

where $X^{[q_b]} = \mathrm{id}^{\otimes(2q_b-2)} \otimes X \otimes \mathrm{id}^{\otimes(2n-2q_b)}$ for $1 \leq b \leq m$. In particular, we have

$$\varphi_{x_1 \to x_n}(\widetilde{A}) \equiv \rho_{x_1 \to x_n}(A) \mod \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2n}[x_1, \ldots, x_n]_{d-1},$$

where $\rho_{x_1 \to x_n}$ is the representation (4.3.1). By Lemma 4.3.1, there exists $a_1, \ldots, a_n \in \mathbb{C}$ such that $\rho_{a_1 \to a_n}(A) \neq 0$; thus, $\varphi_{x_1 \to x_n}(\widetilde{A})$ has polynomial degree $d$, so $\widetilde{A}$ is of filtration degree $d$. □

We now arrive at the Poincaré-Birkhoff-Witt-type theorem for the Yangian as an immediate consequence of Theorem 4.3.2 and the Poincaré-Birkhoff-Witt theorem for Lie superalgebras:

**Corollary 4.3.3** (PBW Theorem for $\mathrm{Y}(\mathfrak{p}_N)$). *Let $\mathcal{B}$ be an index set of pairs $(i, j, n)$ in $(\mathbb{Z}^+)^2 \times \mathbb{N}$ such that $\{\mathsf{E}_{ij}^{(n)} \mid (i, j, n) \in \mathcal{B}\}$ forms a basis for $\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}$. Given any total ordering '$\preceq$' on the set $\mathsf{B} = \{\mathcal{T}_{ij}^{(n+1)} \mid (i, j, n) \in \mathcal{B}\}$, the collection of all ordered monomials of the form*

$$\mathcal{T}_{i_1 j_1}^{(n_1)} \mathcal{T}_{i_2 j_2}^{(n_2)} \cdots \mathcal{T}_{i_k j_k}^{(n_k)},$$

*where $\mathcal{T}_{i_a j_a}^{(n_a)} \in \mathsf{B}$, $\mathcal{T}_{i_a j_a}^{(n_a)} \preceq \mathcal{T}_{i_{a+1} j_{a+1}}^{(n_{a+1})}$, and $\mathcal{T}_{i_a j_a}^{(n_a)} \neq \mathcal{T}_{i_{a+1} j_{a+1}}^{(n_{a+1})}$ if $\mathcal{T}_{i_a j_a}^{(n_a)}$ is odd, constitutes a basis for the Yangian $\mathrm{Y}(\mathfrak{p}_N)$.*

For instance, the index set $\mathcal{B}$ may be constructed via the collection of all tuples $(i, j, n) \in (\mathbb{Z}^+)^2 \times \mathbb{N}$ that satisfying any of the following four conditions:

$$1 \leq |i| < |j| \leq N,\ n \in \mathbb{N}; \quad 1 \leq i = j \leq N,\ n \in \mathbb{N};$$
$$1 \leq -i = j \leq N,\ n \in 2N{+}1; \quad \text{or} \quad -N \leq -i = j \leq -1,\ n \in 2\mathbb{N}.$$

Suppose $\mathfrak{g}$ denotes a Lie superalgebra with trivial supercenter and assume there exists an involution $(-)^{\vartheta} \in \mathrm{Aut}(\mathfrak{g})$. By extending $(-)^{\vartheta}$ to an involutive automorphism of $\mathfrak{g}[z]$ in a similar way to $(-)^{\iota^{\mathsf{P}}}$, i.e., $(X \otimes f(z))^{\vartheta} = X^{\vartheta} \otimes f(-z)$ for $X \in \mathfrak{g}$ and $f(z) \in \mathbb{C}[z]$, then it is a result of [Naz99, Proposition 3.6] that the supercenter of $\mathfrak{U}(\mathfrak{g}[z]^{\vartheta})$ must also be trivial, where $\mathfrak{g}[z]^{\vartheta}$ denotes the fixed-point Lie sub-superalgebra of $\mathfrak{g}[z]$ under the automorphism $(-)^{\vartheta}$. Via this result and the previous theorem, we obtain another corollary:

**Corollary 4.3.4.** *The supercenter* $\mathrm{ZY}(\mathfrak{p}_N)$ *of* $\mathrm{Y}(\mathfrak{p}_N)$ *is trivial:* $\mathbb{C}\cdot\mathbf{1}$.

*Proof.* By the decomposition $\mathfrak{gl}_{N|N} \cong (\mathbb{C}\cdot\mathrm{id}) \oplus (\mathfrak{gl}_{N|N}/\mathbb{C}\cdot\mathrm{id})$, it follows that

$$\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}} \cong (\mathfrak{gl}_{N|N}/\mathbb{C}\cdot\mathrm{id})[z]^{\iota^{\mathsf{P}}},$$

so $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ has trivial supercenter as established by [Naz99, Proposition 3.6]. In particular, the associated graded $\mathrm{gr}\,\mathrm{Y}(\mathfrak{p}_N)$ therefore has trivial supercenter via Theorem 4.3.2, which implies the same is true for $\mathrm{Y}(\mathfrak{p}_N)$ as well. □

**Proposition 4.3.5.** *There is a Hopf superalgebra embedding*

$$\iota\colon \mathfrak{U}(\mathfrak{p}_N) \hookrightarrow \mathrm{Y}(\mathfrak{p}_N), \quad \mathsf{E}_{ij} \mapsto -(-1)^{[i][j]}\mathcal{T}_{ji}^{(1)} \quad \textit{for} \quad i,j \in I_N.$$

*Proof.* Taking the coefficient of $u^{-1}$ in the relations (4.2.29) give

$$\begin{aligned}[\mathcal{T}_{ji}^{(1)}, \mathcal{T}_{lk}(v)] = {} & -\delta_{jk}(-1)^{[i][j]+[i][l]+[k][l]}\mathcal{T}_{li}(v) + \delta_{il}(-1)^{[i]}\mathcal{T}_{jk}(v) \\ & + \delta_{i,-k}(-1)^{[i][l]+[i]+[j]+[j][l]}\mathcal{T}_{l,-j}(v) - \delta_{j,-l}(-1)^{[i][j]}\mathcal{T}_{-i,k}(v),\end{aligned}$$

so one takes the coefficient of $-(-1)^{[i][j]+[k][l]}v^{-1}$ above. Furthermore, we realize that relation (4.2.30) infers $\mathcal{T}_{ji}^{(1)} - (-1)^{[i][j]+[j]}\mathcal{T}_{-i,-j}^{(1)} = 0$, so one multiplies this relation by the scalar $-(-1)^{[i][j]}$. The Hopf superstructures are compatible by their definitions, so all that remains to show is injectivity, but this follows from Corollary 4.3.3. □

### 4.3.3 Homogeneous quantization

When $\mathfrak{g}$ is any finite-dimensional Lie superalgebra, Nazarov described in [Naz99, §1] that given any even, super-symmetric, and $\mathfrak{g}$-invariant element $\omega \in \mathfrak{g}\otimes\mathfrak{g}$, then rational function $r(u,v) = \omega/(u-v)$ is *antisymmetric* $(r(u,v) + \sigma(r(v,u)) = 0)$ and is an *r-matrix*, i.e., $\mathrm{SCYB}(r(u,v)) = 0$, where

$$\begin{aligned}&\mathrm{SCYB}(r(u,v)) \\ &\quad = [r_{12}(u_1,u_2), r_{13}(u_1,u_3)] + [r_{12}(u_1,u_2), r_{23}(u_2,u_3)] + [r_{13}(u_1,u_3), r_{23}(u_2,u_3)].\end{aligned}$$

We refer the reader to §2.3.3 for a more detailed exposition on the notation for the super classical Yang-Baxter equation (SCYBE). In particular, such an element $\omega \in \mathfrak{g}\otimes\mathfrak{g}$

allows one to define a map

$$\begin{aligned}\delta_\omega\colon \mathfrak{g}[z] &\to (\mathfrak{g}\otimes\mathfrak{g})\,[u,v] \cong \mathfrak{g}[z]\otimes\mathfrak{g}[z]\\ f(z) &\mapsto \left(\mathrm{ad}_{f(u)}\otimes \mathrm{id} + \mathrm{id}\otimes \mathrm{ad}_{f(v)}\right)\left(\frac{\omega}{u-v}\right),\end{aligned} \tag{4.3.10}$$

which will be the Lie co-superbracket for a Lie superbialgebra structure $(\mathfrak{g}[z], \delta_\omega)$ on the polynomial current Lie superalgebra $\mathfrak{g}[z]$.

When $\mathfrak{g}$ is basic, such an element $\omega \in \mathfrak{g}\otimes\mathfrak{g}$ exists as one can select it to be the Casimir 2-tensor due to the fact that $\mathfrak{g}$ is equipped with an even, non-degenerate, super-symmetric, and $\mathfrak{g}$-invariant bilinear form (see §2.3.3). With similar reasoning, such an element also exists when $\mathfrak{g} = \mathfrak{gl}_{M|N}$ as the the super trace induces a bilinear form

$$(\,\cdot\,,\cdot\,)\colon \mathfrak{gl}_{M|N}\times\mathfrak{gl}_{M|N}\to\mathbb{C}, \quad (X,Y)\mapsto \mathrm{str}(XY)$$

which is even, non-degenerate, super-symmetric, and $\mathfrak{gl}_{N|N}$-invariant. Hence, taking the Casimir 2-tensor $\Omega$ of $\mathfrak{gl}_{M|N}$ to be the preimage of the identity element in $\mathrm{End}(\mathfrak{gl}_{M|N})$ under the isomorphism

$$\mathfrak{gl}_{M|N}\otimes\mathfrak{gl}_{M|N} \xrightarrow{\sim} \mathfrak{gl}_{M|N}\otimes\mathfrak{gl}^*_{M|N} \xrightarrow{\sim} \mathrm{End}(\mathfrak{gl}_{M|N}),$$

then $\Omega \in \mathfrak{gl}_{M|N}^{\otimes 2}$ satisfies the required properties to define the above Lie superbialgebra structure on $\mathfrak{gl}_{M|N}[z]$. In fact, the Casimir 2-tensor of $\mathfrak{gl}_{M|N}$ takes a familiar form: since the dual basis $\{E^*_{ij}\}_{i,j\in I_N}$ of the standard basis $\{E_{ij}\}_{i,j\in I_N}$ of $\mathfrak{gl}_{N|N}$ with respect to the above bilinear form is given by $E^*_{ij} = (-1)^{[i]}E_{ji}$, the element $\Omega$ can be written as

$$\Omega = \sum_{i,j\in I_N}(-1)^{[E_{ij}]}E_{ij}\otimes(-1)^{[i]}E_{ji} = \sum_{i,j\in I_N}(-1)^{[j]}E_{ij}\otimes E_{ji},$$

which is the super permutation operator $P$. By the $\mathfrak{gl}_{M|N}$-invariance of $P$, the Lie co-superbracket on $\mathfrak{gl}_{M|N}[z]$ is equivalent to the assignment

$$\delta_P(Xz^n) = \sum_{i,j\in I_N}\sum_{a=0}^{n-1}(-1)^{[j]}[X,E_{ij}]z^a\otimes E_{ji}z^{n-a-1}$$

for all $X\in\mathfrak{gl}_{M|N}$ and $n\in\mathbb{Z}^+$, where it is understood that $\delta_P(X)=0$.

A natural question to ask is whether or not it is possible to define such a Lie superbialgebra structure on the strange Lie superalgebras $\mathfrak{s}_N = \mathfrak{p}_N, \mathfrak{q}_N$ in a similar way via the use of an appropriate element in $\mathfrak{s}_N^{\otimes 2}$. However, as was observed by M. Nazarov in [Naz92], the above map $\delta_\omega$ fails to be non-trivial in these cases due to the fact that all even, super-symmetric, and $\mathfrak{s}_N$-invariant elements $\omega \in \mathfrak{s}_N^{\otimes 2}$ are trivial:

$$\omega \in \mathbb{C}\textstyle\sum_{i,j=1}^N \mathsf{E}_{ii} \otimes \mathsf{E}_{jj} \text{ if } \mathfrak{s}_N = \mathfrak{p}_N \qquad \text{and} \qquad \omega \in \mathbb{C}\,\mathrm{id}^{\otimes 2} \text{ if } \mathfrak{s}_N = \mathfrak{q}_N.$$

In either case, it is verifiable to check $\delta_\omega = 0$. In light of this, Nazarov observed that one can instead define a non-trivial Lie superbialgebra structure on the twisted polynomial current Lie superalgebras $\mathfrak{gl}_{M|N}[z]^{\iota^K}$ for $K = \mathsf{P}, \mathsf{Q}$, so we will state such construction here. To start, it is proven by [Naz99, Proposition 1.1] that the rational function

$$r^K(u,v) = \frac{P}{u-v} + \frac{Q^K}{u+v} \in \mathfrak{gl}_{N|N}^{\otimes 2}(u,v) \tag{4.3.11}$$

is antisymmetric and an $r$-matrix ($\mathrm{SCYB}(r^K(u,v)) = 0$); in particular, such $r$-matrix allows one to define a map

$$\begin{aligned} \delta_K \colon \mathfrak{gl}_{N|N}[z] &\to \left(\mathfrak{gl}_{N|N} \otimes \mathfrak{gl}_{N|N}\right)[u,v] \cong \mathfrak{gl}_{N|N}[z] \otimes \mathfrak{gl}_{N|N}[z] \\ f(z) &\mapsto \left(\mathrm{ad}_{f(u)} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{f(v)}\right)\left(r^K(u,v)\right). \end{aligned}$$

In fact, since $(\mathrm{id} \otimes \mathrm{ad}_{f(v)})(Q^K/(u+v)) = \left((\mathrm{id} \otimes \mathrm{ad}_{f(-v)})(P/(u+v))\right)^{\iota_2^K}$ for polynomials $f(z) \in \mathfrak{gl}_{N|N}[z]^{\iota^K}$, the $\mathfrak{gl}_{N|N}$-invariance of $P$ implies that the function $\delta$ restricts to well-defined map $\delta \colon \mathfrak{gl}_{N|N}[z]^{\iota^K} \to \mathfrak{gl}_{N|N}[z]^{\iota^K} \otimes \mathfrak{gl}_{N|N}[z]^{\iota^K}$ given by

$$\delta_K(Xz^n) = \sum_{i,j \in I_N} \sum_{a+b=n-1} (-1)^{[j]}[X, E_{ij}]z^a \otimes (E_{ji}z^b + E_{ji}^{\iota^K}(-z)^b), \tag{4.3.12}$$

where $X$ has polynomial degree 0 and it is understood that $\delta(X) = 0$. In terms of the generators of $\mathfrak{gl}_{N|N}[z]^{\iota^K}$, one can compute the formula

$$\delta_{\mathsf{P}}(\mathsf{E}_{ij}^{(n)}(z)) = \sum_{k \in I_N} (-1)^{[k]} \sum_{a+b=n-1} \left(\mathsf{E}_{ik}^{(a)}(z) \otimes \mathsf{E}_{kj}^{(b)}(z) - (-1)^{([i]+[k])([k]+[j])}\mathsf{E}_{kj}^{(a)}(z) \otimes \mathsf{E}_{ik}^{(b)}(z)\right)$$

for $i, j \in I_N$ and $n \in \mathbb{Z}^+$ when $K = \mathsf{P}$, with a similar formula holding when $K = \mathsf{Q}$ (see [Naz99, §2]). Hence, it follows from the properties of the $r$-matrix $r^K(u,v)$ that $\delta_K$ becomes a Lie co-superbracket for a Lie superbialgebra structure $(\mathfrak{gl}_{N|N}[z]^{\iota^K}, \delta_K)$ on the space $\mathfrak{gl}_{N|N}[z]^{\iota^K}$.

We adopt the same definitions of (homogeneous) Hopf superalgebra deformations and quantizations as in §2.3.3; namely, we assume such notions are taken over the polynomial ring $\mathbb{C}[\hbar]$, where $\hbar$ is a formal element of $\mathbb{Z}_2$-degree $\bar{0}$. As noted in Chapter 2, if $\mathbb{U}_\hbar(\mathfrak{b})$ is any Hopf superalgebra deformation of $\mathfrak{U}(\mathfrak{b})$ for any Lie superalgebra $\mathfrak{b}$, then $\mathfrak{b}$ is endowed with a Lie superbialgebra structure $(\mathfrak{b}, \delta_\mathfrak{b})$ defined by the Lie co-superbracket

$$\delta_\mathfrak{b}(X) := \frac{\Delta_\hbar(\widetilde{X}) - \Delta_\hbar^{\mathrm{cop}}(\widetilde{X})}{\hbar} \mod \hbar\,(\mathbb{U}_\hbar(\mathfrak{b}) \otimes \mathbb{U}_\hbar(\mathfrak{b})) \qquad \text{for all} \quad X \in \mathfrak{b}, \tag{4.3.13}$$

where $\Delta_\hbar$ is the comultiplication map on $\mathbb{U}_\hbar(\mathfrak{b})$, $\Delta_\hbar^{\mathrm{cop}} = \sigma \circ \Delta_\hbar$ is the co-opposite comultiplication, and $\widetilde{X}$ is any element in the fiber of $X \in \mathfrak{b} \hookrightarrow \mathfrak{U}(\mathfrak{b})$ under the composition $\mathbb{U}_\hbar(\mathfrak{b}) \twoheadrightarrow \mathbb{U}_\hbar(\mathfrak{b})/\hbar\mathbb{U}_\hbar(\mathfrak{b}) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{b})$.

Furthermore, as discussed in [Wen22], if $\mathbb{U}_\hbar(\mathfrak{b})$ is a homogeneous quantization (over $\mathbb{C}[\hbar]$) of an $\mathbb{N}$-graded Lie superbialgebra $(\mathfrak{b}, \delta_\mathfrak{b})$, then its $\hbar$-adic completion

$$\widehat{\mathbb{U}}_\hbar(\mathfrak{b}) = \varprojlim \mathbb{U}_\hbar(\mathfrak{b})/\hbar^n\mathbb{U}_\hbar(\mathfrak{b})$$

will be a homogeneous quantization of $(\mathfrak{b}, \delta_\mathfrak{b})$ in the sense of [Dri85], taking into account the super-analogues of the definitions therein. We shall now construct such a homogeneous quantization of $(\mathfrak{gl}_{N|N}[z]^{\iota^K}, \delta_K)$, where $\delta_K$ is the Lie co-superbracket (4.3.12).

**Definition 4.3.6.** Let $\mathfrak{s}_N$ denote either $\mathfrak{p}_N$ or $\mathfrak{q}_N$. Given $\mathbb{C}[\hbar] \otimes \mathrm{Y}(\mathfrak{s}_N) = \mathrm{Y}(\mathfrak{s}_N)[\hbar]$ where $\hbar$ is a formal element of $\mathbb{Z}_2$-degree $\bar{0}$, the *Yangian* $\mathrm{Y}_\hbar(\mathfrak{s}_N)$ is defined as the Rees superalgebra of $\mathrm{Y}(\mathfrak{s}_N)$ with respect to the filtration $\mathbf{F}(\mathrm{Y}(\mathfrak{s}_N)) = \{\mathbf{F}_n(\mathrm{Y}(\mathfrak{s}_N))\}_{n\in\mathbb{N}}$ on $\mathrm{Y}(\mathfrak{s}_N)$ defined by the assignment $\deg \mathcal{T}_{ij}^{(n)} = n-1$:

$$\mathrm{Y}_\hbar(\mathfrak{s}_N) := R_\hbar(\mathrm{Y}(\mathfrak{s}_N)) = \bigoplus_{n\in\mathbb{N}} \hbar^n \mathbf{F}_n(\mathrm{Y}(\mathfrak{s}_N)) \subset \mathrm{Y}(\mathfrak{s}_N)[\hbar].$$

By definition, the Yangian $\mathrm{Y}_\hbar(\mathfrak{s}_N)$ is $\mathbb{N}$-graded and it further comes equipped with a Hopf superstructure by extending the one on $\mathrm{Y}(\mathfrak{s}_N)$ by $\mathbb{C}[\hbar]$-linearity. In particular, by setting $\widetilde{\mathcal{T}}_{ij}^{(n)} = \hbar^{n-1}\mathcal{T}_{ij}^{(n)}$ for all $i, j \in I_N$ and $n \in \mathbb{Z}^+$, such Hopf superstructure is given by the comultiplication

$$\begin{aligned} \Delta_\hbar \colon \ & \mathrm{Y}_\hbar(\mathfrak{s}_N) \to \mathrm{Y}_\hbar(\mathfrak{s}_N) \otimes_{\mathbb{C}[\hbar]} \mathrm{Y}_\hbar(\mathfrak{s}_N) \\ & \widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto \widetilde{\mathcal{T}}_{ij}^{(n)} \otimes \mathbf{1} + \mathbf{1} \otimes \widetilde{\mathcal{T}}_{ij}^{(n)} + \hbar \textstyle\sum_{k=1}^{M+N} \sum_{a=1}^{n-1} \widetilde{\mathcal{T}}_{ik}^{(a)} \otimes \widetilde{\mathcal{T}}_{kj}^{(n-a)}, \end{aligned}$$

the counit

$$\varepsilon_\hbar\colon \mathrm{Y}_\hbar(\mathfrak{s}_N) \to \mathbb{C}[\hbar], \quad \widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto 0,$$

for all $i, j \in I_N$ and $n \in \mathbb{Z}^+$, whilst the antipode

$$S_\hbar\colon \mathrm{Y}_\hbar(\mathfrak{s}_N) \to \mathrm{Y}_\hbar(\mathfrak{s}_N)$$

is given by the assignment

$$\widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto -\widetilde{\mathcal{T}}_{ij}^{(n)} + \sum_{s=2}^{n} (-1)^s \hbar^{s-1} \sum_{\sum_{j=1}^{s} k_j = n} \left( \sum_{a_1, a_2, \dots, a_{s-1} \in I_N} \widetilde{\mathcal{T}}_{ia_1}^{(k_1)} \widetilde{\mathcal{T}}_{a_1 a_2}^{(k_2)} \cdots \widetilde{\mathcal{T}}_{a_{s-1} j}^{(k_s)} \right)$$

with $k_j \in \mathbb{Z}^+$ for each term in the sum $\sum_{j=1}^{s} k_j = n$ and $i, j \in I_N$, $n \in \mathbb{Z}^+$. When $\mathfrak{s}_N = \mathfrak{p}_N$, we note the antipode takes on the simpler form $S_\hbar(\widetilde{\mathcal{T}}_{ij}^{(n)}) = (-1)^{[i][j]+[i]+n} \widetilde{\mathcal{T}}_{-j,-i}^{(n)}$. We now arrive at the main proposition of this subsection, which was first stated in [Naz92]. Note that the proof of the following proposition is completely analogous to the proof of Proposition 2.3.8.

**Proposition 4.3.7.** *The Yangian $\mathrm{Y}_\hbar(\mathfrak{p}_N)$ is a homogeneous quantization of the Lie superbialgebra $(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}, \delta_{\mathsf{P}})$, whilst the Yangian $\mathrm{Y}_\hbar(\mathfrak{q}_N)$ is a homogeneous quantization of the Lie superbialgebra $(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{Q}}}, \delta_{\mathsf{Q}})$. Furthermore, there is a superalgebra isomorphism*

$$\mathrm{Y}_\hbar(\mathfrak{s}_N)/(\hbar - \lambda)\,\mathrm{Y}_\hbar(\mathfrak{s}_N) \cong \mathrm{Y}(\mathfrak{s}_N) \quad \textit{for all} \quad \lambda \in \mathbb{C}^*,$$

*where $\mathfrak{s}_N$ denotes either $\mathfrak{p}_N$ or $\mathfrak{q}_N$.*

*Proof.* We shall provide the proof for $\mathfrak{s}_N = \mathfrak{p}_N$ since the case $\mathfrak{s}_N = \mathfrak{q}_N$ is similar with much of its proof already provided in [Naz99, Proposition 2.5]. To show $\mathrm{Y}_\hbar(\mathfrak{p}_N)$ is a homogeneous Hopf superalgebra deformation of $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$, we initially observe that $\mathrm{Y}_\hbar(\mathfrak{p}_N)$ is torsion-free, being a $\mathbb{C}[\hbar]$-subalgebra of $\mathrm{Y}(\mathfrak{p}_N)[\hbar]$. In particular, the composition of the Hopf superalgebra isomorphism

$$\phi\colon \mathrm{Y}_\hbar(\mathfrak{p}_N)/\hbar\,\mathrm{Y}_\hbar(\mathfrak{p}_N) \xrightarrow{\sim} \operatorname{gr}\mathrm{Y}(\mathfrak{p}_N), \quad \hbar^{n-1}\mathcal{T}_{ij}^{(n)} \bmod \hbar\,\mathrm{Y}_\hbar(\mathfrak{p}_N) \mapsto \overline{\mathcal{T}}_{ij}^{(n)}$$

for $i, j \in I_N$, $n \in \mathbb{Z}^+$, with the inverse of the isomorphism $\Phi$ (4.3.6) gives the desired

$\mathbb{N}$-graded Hopf superalgebra isomorphism

$$\Phi^{-1} \circ \phi \colon \mathrm{Y}_\hbar(\mathfrak{p}_N)/\hbar\,\mathrm{Y}_\hbar(\mathfrak{p}_N) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}).$$

By the prior discussion, it follows that $\mathrm{Y}_\hbar(\mathfrak{p}_N)$ homogeneously quantizes the Lie superbialgebra structure on $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ with Lie co-superbracket given by (4.3.13). In fact, such Lie co-superbracket coincides with the one given by (4.3.12) with $K = \mathsf{P}$, since defining $\mathrm{ev}_\hbar$ as the morphism

$$\mathrm{ev}_\hbar \colon \mathrm{Y}_\hbar(\mathfrak{p}_N) \twoheadrightarrow \mathrm{Y}_\hbar(\mathfrak{p}_N)/\hbar\,\mathrm{Y}_\hbar(\mathfrak{p}_N) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$$

mapping $\hbar^{n-1}\mathcal{T}_{ij}^{(n)} \mapsto -(-1)^{[i][j]}\mathsf{E}_{ji}^{(n-1)}(z)$ for $i, j \in I_N$ and $n \in \mathbb{Z}^+$, we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{Y}_\hbar(\mathfrak{p}_N) & \xrightarrow{\hbar^{-1}(\Delta_\hbar - \Delta_\hbar^{\mathrm{cop}})} & \mathrm{Y}_\hbar(\mathfrak{p}_N)^{\otimes 2} \\ {\scriptstyle \mathrm{ev}_\hbar}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_\hbar \otimes \mathrm{ev}_\hbar} \\ \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) & \xrightarrow[\delta_{\mathsf{P}}]{} & \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})^{\otimes 2} \end{array}$$

where $\delta_{\mathsf{P}}$ is the extension of the Lie co-superbracket (4.3.12) to a coPoisson superbracket on $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$.

For the second claim, we consider the epimorphism $\mathrm{ev}_\lambda \colon \mathrm{Y}(\mathfrak{p}_N)[\hbar] \to \mathrm{Y}(\mathfrak{p}_N)$ induced by the assignment $\hbar \mapsto \lambda$. The restriction $\mathrm{ev}_\lambda^R$ of $\mathrm{ev}_\lambda$ to $R_\hbar(\mathrm{Y}(\mathfrak{p}_N))$ will still remain surjective and its kernel is given by

$$\ker(\mathrm{ev}_\lambda^R) = R_\hbar(\mathrm{Y}(\mathfrak{p}_N)) \cap (\hbar - \lambda)\,\mathrm{Y}(\mathfrak{p}_N)[\hbar] = (\hbar - \lambda) R_\hbar(\mathrm{Y}(\mathfrak{p}_N)),$$

which finishes the proof. □

As discussed earlier in this subsection, it therefore follows by the work in [Wen22] that the $\hbar$-adic completions

$$\widehat{\mathrm{Y}}_\hbar(\mathfrak{p}_N) = \varprojlim \mathrm{Y}_\hbar(\mathfrak{p}_N)/\hbar^n\,\mathrm{Y}_\hbar(\mathfrak{p}_N) \quad \text{and} \quad \widehat{\mathrm{Y}}_\hbar(\mathfrak{q}_N) = \varprojlim \mathrm{Y}_\hbar(\mathfrak{q}_N)/\hbar^n\,\mathrm{Y}_\hbar(\mathfrak{q}_N)$$

are homogeneous quantizations of $(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}, \delta_{\mathsf{P}})$ and $(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{Q}}}, \delta_{\mathsf{Q}})$, respectively, in the sense of [Dri85].

The remainder of this subsection is devoted to expressing the Yangian $\mathrm{Y}_\hbar(\mathfrak{s}_N)$ in terms of generators and relations for $\mathfrak{s}_N = \mathfrak{p}_N, \mathfrak{q}_N$. Similar to Chapter 2, we will define certain $\mathbb{C}[\hbar]$-superalgebras $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$ and $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{q}_N)$ built on generators subject to relations and ultimately show that these are respectively isomorphic to the Yangians $\mathrm{Y}_\hbar(\mathfrak{p}_N)$ and $\mathrm{Y}_\hbar(\mathfrak{q}_N)$.

**Definition 4.3.8.** Define $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$ as the unital associative $\mathbb{C}[\hbar]$-superalgebra on the generators $\{\widetilde{\mathcal{T}}_{ij}^{(n)} \mid i,j \in I_N,\, n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\big[\widetilde{\mathcal{T}}_{ij}^{(n)}\big] = [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the relations

$$\begin{aligned}\big[\widetilde{\mathcal{T}}_{ij}^{(m)}, \widetilde{\mathcal{T}}_{kl}^{(n)}\big] = \delta_{jk}(-1)^{[k]}\widetilde{\mathcal{T}}_{il}^{(m+n-1)} &- \delta_{il}(-1)^{[i][k]+[j][k]+[j][l]}\widetilde{\mathcal{T}}_{kj}^{(m+n-1)} \\ + \delta_{i,-k}(-1)^{[i][j]+m}\widetilde{\mathcal{T}}_{-j,l}^{(m+n-1)} &- \delta_{j,-l}(-1)^{[i][k]+[j][k]+[j]+m}\widetilde{\mathcal{T}}_{k,-i}^{(m+n-1)} \\ + (-1)^{[i][j]+[i][k]+[j][k]}\hbar &\sum_{a=2}^{\min(m,n)} \big(\widetilde{\mathcal{T}}_{kj}^{(a-1)}\widetilde{\mathcal{T}}_{il}^{(m+n-a)} - \widetilde{\mathcal{T}}_{kj}^{(m+n-a)}\widetilde{\mathcal{T}}_{il}^{(a-1)}\big) \\ - \delta_{i,-k}\,\hbar &\sum_{a=2}^{m}\sum_{p\in I_N}(-1)^{[i][j]+[j][p]+[p]+m-a}\widetilde{\mathcal{T}}_{pj}^{(a-1)}\widetilde{\mathcal{T}}_{-p,l}^{(m+n-a)} \\ + \delta_{j,-l}\,\hbar &\sum_{a=2}^{m}\sum_{p\in I_N}(-1)^{[i][k]+[i]+[j][k]+[j]+[i][p]+m-a}\widetilde{\mathcal{T}}_{k,-p}^{(m+n-a)}\widetilde{\mathcal{T}}_{ip}^{(a-1)}\end{aligned}$$

and

$$(-1)^{[i][j]+[i]}\widetilde{\mathcal{T}}_{-j,-i}^{(n)} = (-1)^n\widetilde{\mathcal{T}}_{ij}^{(n)} + \hbar\sum_{p\in I_N}\sum_{a=1}^{n-1}(-1)^{[i][p]+[i]+n-a-1}\widetilde{\mathcal{T}}_{-p,-i}^{(a)}\widetilde{\mathcal{T}}_{pj}^{(n-a)}$$

for all $i,j,k,l \in I_N$ and $m,n \in \mathbb{Z}^+$.

**Definition 4.3.9.** Define $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{q}_N)$ as the unital associative $\mathbb{C}[\hbar]$-superalgebra on the generators $\{\widetilde{\mathcal{T}}_{ij}^{(n)} \mid i,j \in I_N,\, n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\big[\widetilde{\mathcal{T}}_{ij}^{(n)}\big] = [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the relations

$$\begin{aligned}&(-1)^{[i][j]+[i][k]+[j][k]}\big[\widetilde{\mathcal{T}}_{ij}^{(m)}, \widetilde{\mathcal{T}}_{kl}^{(n)}\big] \\ &\quad= \delta_{jk}\widetilde{\mathcal{T}}_{il}^{(m+n-1)} - \delta_{il}\widetilde{\mathcal{T}}_{kj}^{(m+n-1)} - \delta_{j,-k}(-1)^m\widetilde{\mathcal{T}}_{-i,l}^{(m+n-1)} + \delta_{i,-l}(-1)^m\widetilde{\mathcal{T}}_{k,-j}^{(m+n-1)} \\ &\qquad+ \hbar\sum_{a=2}^{\min(m,n)}\big(\widetilde{\mathcal{T}}_{kj}^{(a-1)}\widetilde{\mathcal{T}}_{il}^{(m+n-a)} - \widetilde{\mathcal{T}}_{kj}^{(m+n-a)}\widetilde{\mathcal{T}}_{il}^{(a-1)}\big) \\ &\qquad- \hbar\sum_{a=2}^{m}(-1)^{m-a}\Big((-1)^{[j]+[k]}\widetilde{\mathcal{T}}_{-k,j}^{(a-1)}\widetilde{\mathcal{T}}_{-i,l}^{(m+n-a)} - (-1)^{[i]+[l]}\widetilde{\mathcal{T}}_{k,-j}^{(m+n-a)}\widetilde{\mathcal{T}}_{i,-l}^{(a-1)}\Big)\end{aligned}$$

$$\text{and} \qquad \widetilde{\mathcal{T}}_{-i,-j}^{(n)} = (-1)^{[i]+[j]+n}\widetilde{\mathcal{T}}_{ij}^{(n)}$$

for all $i, j, k, l \in I_N$ and $m, n \in \mathbb{Z}^+$.

For $\mathfrak{s}_N = \mathfrak{p}_N, \mathfrak{q}_N$, we observe the superalgebra $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{s}_N)$ is $\mathbb{N}$-graded via the gradation assignments

$$\deg \hbar = 1 \quad \text{and} \quad \deg \widetilde{\mathcal{T}}_{ij}^{(n)} = n-1 \quad \text{for} \quad i, j \in I_N,\ n \in \mathbb{Z}^+.$$

In Proposition 4.3.11 below, we will show $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{s}_N) \cong \mathrm{Y}_\hbar(\mathfrak{s}_N)$. Again, the following arguments are completely analogous to those used in §2.3.3, which themselves derive from the articles [GRW19a, Proposition 2.2] and [GRW19c, Theorem 6.10].

For $K = \mathsf{P}, \mathsf{Q}$, by equipping $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^K})$ with a $\mathbb{C}[\hbar]$-superalgebra structure via the action induced by $\hbar \mapsto 0$, we get the following result:

**Lemma 4.3.10.** *There are* $\mathbb{N}$*-graded superalgebra epimorphisms*

$$\widetilde{\mathrm{ev}}_\hbar^{\mathsf{P}} \colon \widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N) \twoheadrightarrow \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \quad \text{and} \quad \widetilde{\mathrm{ev}}_\hbar^{\mathsf{Q}} \colon \widetilde{\mathrm{Y}}_\hbar(\mathfrak{q}_N) \twoheadrightarrow \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{Q}}})$$

*defined by* $\widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto -(-1)^{[i][j]}\mathsf{E}_{ji}^{(n-1)}(z)$ *and* $\widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto -(-1)^{[i][j]}\mathsf{F}_{ji}^{(n-1)}(z)$*, respectively, for all* $i, j \in I_N$, $n \in \mathbb{Z}^+$*. In particular,* $\ker(\widetilde{\mathrm{ev}}_\hbar^{\mathsf{P}}) = \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$ *and* $\ker(\widetilde{\mathrm{ev}}_\hbar^{\mathsf{Q}}) = \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{q}_N)$*, so there are isomorphisms*

$$\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)/\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N) \cong \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \quad \text{and} \quad \widetilde{\mathrm{Y}}_\hbar(\mathfrak{q}_N)/\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{q}_N) \cong \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{Q}}})$$

*as* $\mathbb{N}$*-graded superalgebras.*

*Proof.* We shall provide the proof for $\mathfrak{s}_N = \mathfrak{p}_N$, and the case $\mathfrak{s}_N = \mathfrak{q}_N$ is similar. By the $\mathbb{C}[\hbar]$-module structure on $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$, it is routine to prove $\widetilde{\mathrm{ev}}_\hbar^{\mathsf{P}}$ is a gradation preserving superalgebra epimorphism such that $\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N) \subseteq \ker(\widetilde{\mathrm{ev}}_\hbar^{\mathsf{P}})$; hence, $\widetilde{\mathrm{ev}}_\hbar^{\mathsf{P}}$ descends to an epimorphism $\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)/\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N) \to \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ of $\mathbb{N}$-graded superalgebras mapping $\widetilde{\mathcal{T}}_{ij}^{(n)} \bmod \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N) \mapsto -(-1)^{[i][j]}\mathsf{E}_{ji}^{(n-1)}(z)$. Conversely, there is a superalgebra morphism $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) \to \widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)/\hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$ sending $\mathsf{E}_{ji}^{(n-1)}(z)$ to the element $-(-1)^{[i][j]}\widetilde{\mathcal{T}}_{ij}^{(n)} \bmod \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$, which establishes the isomorphism. □

**Proposition 4.3.11.** *For $\mathfrak{s}_N = \mathfrak{p}_N, \mathfrak{q}_N$, there is an isomorphism of $\mathbb{C}[\hbar]$-superalgebras*

$$\varphi_\hbar \colon \widetilde{\mathrm{Y}}_\hbar(\mathfrak{s}_N) \to \mathrm{Y}_\hbar(\mathfrak{s}_N), \quad \widetilde{\mathcal{T}}_{ij}^{(n)} \mapsto \hbar^{n-1}\mathcal{T}_{ij}^{(n)}$$

*for all $i, j \in I_N$, $n \in \mathbb{Z}^+$.*

*Proof.* We shall provide the proof for $\mathfrak{s}_N = \mathfrak{p}_N$, and the case $\mathfrak{s}_N = \mathfrak{q}_N$ is similar. By the defining relations in the Yangian $\mathrm{Y}(\mathfrak{p}_N)$ and the fact that the elements $\hbar^{n-1}\mathcal{T}_{ij}^{(n)}$, $i, j \in I_N$, $n \in \mathbb{Z}^+$, generate $\mathrm{Y}_\hbar(\mathfrak{p}_N)$, the map $\varphi_\hbar$ is a superalgebra epimorphism. Recalling the $\mathbb{C}[\hbar]$-superalgebra structure on $\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ defined by $\hbar \mapsto 0$, there is an epimorphism $\mathrm{ev}_\hbar^{\mathsf{P}} \colon \mathrm{Y}_\hbar(\mathfrak{p}_N) \twoheadrightarrow \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$ of $\mathbb{C}[\hbar]$-superalgebras induced by $\mathrm{Y}_\hbar(\mathfrak{p}_N)/\hbar\,\mathrm{Y}_\hbar(\mathfrak{p}_N) \cong \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})$. In fact, we have the commuting diagram:

$$\begin{array}{ccc}
\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N) & \xrightarrow{\quad\varphi_\hbar\quad} & \mathrm{Y}_\hbar(\mathfrak{p}_N) \\
{\scriptstyle \widetilde{\mathrm{ev}}_\hbar^{\mathsf{P}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_\hbar^{\mathsf{P}}} \\
\mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}}) & \xrightarrow[\quad\mathrm{id}\quad]{} & \mathfrak{U}(\mathfrak{gl}_{N|N}[z]^{\iota^{\mathsf{P}}})
\end{array}$$

Suppose $X \in \widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$ is nonzero such that $X \in \ker\varphi_\hbar$. As there exists a maximal integer $n \in \mathbb{N}$ such that $X \in \hbar^n\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$, one can write $X = \hbar^n Y$ for some $Y \notin \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$. In particular, since $0 = \varphi_\hbar(\hbar^n Y) = \hbar^n\varphi_\hbar(Y)$, it must be $Y \in \ker\varphi_\hbar$ as well due to $\mathrm{Y}_\hbar(\mathfrak{p}_N)$ being torsion-free. However, the above commutative diagram would imply $Y \in \ker(\widetilde{\mathrm{ev}}_\hbar^{\mathsf{P}}) = \hbar\widetilde{\mathrm{Y}}_\hbar(\mathfrak{p}_N)$, a contradiction. □

# Chapter 5

# Twisted Super Yangians of Type AIII

In this penultimate chapter, we define twisted Yangians associated to symmetric superpairs of type AIII which take the form

$$(\mathfrak{gl}_{M|N}, \mathfrak{gl}_{p|q} \oplus \mathfrak{gl}_{(M-p)|(N-q)}) \quad \text{for} \quad 0 \le p \le M,\ 0 \le q < N.$$

In particular, the first section starts by introducing a family of reflection superalgebras in §5.1.1 that are subject to an additional unitary condition. Such twisted super Yangians are defined in § 5.1.2, where it is shown that they are in fact isomorphic to these reflection superalgebras. In §5.2, a highest weight theory for the super twisted Yangians is cultivated where it is proven that every finite-dimensional irreducible representation must be highest weight.

## 5.1 The Twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$

We shall prove twisted super Yangians of type AIII are also reflection superalgebras subject to an additional unitary condition. As a consequence, we will be able to to establish a PBW-type theorem for these twisted super Yangians by making use of certain properties of the Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N})$.

### 5.1.1 Reflection superalgebras

Given a integers $M, N \in \mathbb{N}$ such that $M+N \geq 1$, we recall the *gradation index* from subsection §2.1.2 when $\mathbf{d} = \{1, 2, \dots, M\}$:

$$[\cdot]\colon \{1, 2, \dots, M+N\} \to \mathbb{Z}_2$$

given by $[i] = \bar{0}$ if $1 \leq i \leq M$ and $[i] = \bar{1}$ if $M+1 \leq i \leq M+N$.

We recall that $\mathbb{C}^{M|N}$ denotes the vector space $\mathbb{C}^{M+N}$ equipped with the $\mathbb{Z}_2$-grading by assigning $[e_i] = [i]$, where $\{e_i\}_{i=1}^{M+N}$ is the standard ordered basis of $\mathbb{C}^{M+N}$. The space of $\mathbb{C}$-linear maps $\mathbb{C}^{M|N} \to \mathbb{C}^{M|N}$, denoted $\operatorname{End}\mathbb{C}^{M|N}$, carries the natural $\mathbb{Z}_2$-grading such that $[E_{ij}] := [i]+[j]$, where $\{E_{ij}\}_{i,j=1}^{M+N}$ is the collection of the matrix units of $\operatorname{End}\mathbb{C}^{M|N}$ with respect to the standard basis. The space $\operatorname{End}\mathbb{C}^{M|N}$ is denoted $\mathfrak{gl}_{M|N} = \mathfrak{gl}(\mathbb{C}^{M|N})$ when given the Lie superalgebra structure via the super-commutator $[E_{ij}, E_{kl}] := \delta_{jk}E_{il} - (-1)^{([i]+[j])([k]+[l])}\delta_{li}E_{kj}$. We also recall the super permutation operator (2.2.1) in $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$ given by

$$P := \sum\nolimits_{i,j=1}^{M+N} (-1)^{[j]} E_{ij} \otimes E_{ji},$$

and the super-transpose (2.1.7) map

$$(-)^{st}\colon \operatorname{End}\mathbb{C}^{M|N} \to \operatorname{End}\mathbb{C}^{M|N}, \quad E_{ij} \mapsto E_{ij}^{st} := (-1)^{[i][j]+[i]} E_{ji}.$$

Throughout Chapter 5, we define the *R-matrix* $R(u)$ to be the rational function in the formal parameter $u$ taking coefficients in $\left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}$ given by

$$R(u) := \mathrm{id}^{\otimes 2} - Pu^{-1}, \tag{5.1.1}$$

which is the simplest non-trivial solution to the *super quantum Yang-Baxter equation* (SQYBE):

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u), \tag{5.1.2}$$

Moreover, the equations $P^2 = \mathrm{id}^{\otimes 2}$ and $P^{st_1 \circ st_2} = P = P^{st_2 \circ st_1}$ infer the relations

$$PR(u)P = R(u) = R^{st_2 \circ st_1}(u) = R^{st_1 \circ st_2}(u), \tag{5.1.3}$$

$$R(u)R(-u) = \left(1 - \frac{1}{u^2}\right) \mathrm{id}^{\otimes 2}, \tag{5.1.4}$$

known as *crossing symmetry* and *unitarity*, respectively.

Consider now integers $M,N\in\mathbb{N}$ such that $M+N\geq 1$. By decomposing $M$ and $N$ into sums of two non-negative integers: $M=p+k_M$ and $N=q+k_N$, we define a matrix $\mathcal{G}=(\mathcal{G}_{ij})_{i,j=1}^{M+N}\in\mathrm{Mat}_{M+N}(\mathbb{C})$ via the formula

$$\mathcal{G}_{ij}:=\begin{cases}\delta_{ij} & \text{if } 1\leq i\leq p \text{ or } M+1\leq i\leq M+q,\\ -\delta_{ij} & \text{if } p+1\leq i\leq M \text{ or } M+q+1\leq i\leq M+N.\end{cases}\tag{5.1.5}$$

The matrix $\mathcal{G}$ will underlie several constructions in this chapter, including the *reflection superalgebras.* These superalgebras are important in twisted Yangian theory and these shall be the first objects we investigate.

**Definition 5.1.1.** The *extended reflection superalgebra* $\mathrm{XB}(\mathfrak{gl}_{M|N},\mathcal{G})$ of $\mathfrak{gl}_{M|N}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{\mathsf{B}_{ij}^{(n)}\mid 1\leq i,j\leq M+N,\ n\in\mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $[\mathsf{B}_{ij}^{(n)}]:=[i]+[j]$ for all $n\in\mathbb{Z}^+$, subject to the defining *super reflection equation*

$$\begin{aligned}R(u-v)\mathsf{B}_1(u)R(u+v)\mathsf{B}_2(v)&=\mathsf{B}_2(v)R(u+v)\mathsf{B}_1(u)R(u-v)\\ \text{in}\quad&\left(\mathrm{End}\,\mathbb{C}^{M|N}\right)^{\otimes 2}\otimes\mathrm{XB}(\mathfrak{gl}_{M|N},\mathcal{G})[\![u^{\pm1},v^{\pm1}]\!],\end{aligned}\tag{5.1.6}$$

where $\mathsf{B}(u):=\sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij}\otimes\mathsf{B}_{ij}(u)\in\mathrm{End}(\mathbb{C}^{M|N})\otimes\mathrm{XB}(\mathfrak{gl}_{M|N},\mathcal{G})[\![u^{-1}]\!]$ is the matrix consisting of the series $\mathsf{B}_{ij}(u):=\mathcal{G}_{ij}\mathbf{1}+\sum_{n=1}^{\infty}\mathsf{B}_{ij}^{(n)}u^{-n}\in\mathrm{XB}(\mathfrak{gl}_{M|N},\mathcal{G})[\![u^{-1}]\!]$ for indices $1\leq i,j\leq M+N$, and $R(u-v)$ is the $R$-matrix (5.1.1) identified with $R(u-v)\otimes\mathbf{1}$.

On the level of power series, the super reflection equation (5.1.6) takes the form

$$\begin{aligned}\left[\mathsf{B}_{ij}(u),\mathsf{B}_{kl}(v)\right]&=\frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\left(\mathsf{B}_{kj}(u)\mathsf{B}_{il}(v)-\mathsf{B}_{kj}(v)\mathsf{B}_{il}(u)\right)\\&\quad+\frac{(-1)^{[j][k]}}{u+v}\Big(\delta_{jk}\sum_{a=1}^{M+N}\mathsf{B}_{ia}(u)\mathsf{B}_{al}(v)-(-1)^{[i][j]+[i][k]}\delta_{il}\sum_{a=1}^{M+N}\mathsf{B}_{ka}(v)\mathsf{B}_{aj}(u)\Big)\\&\quad-\frac{1}{u^2-v^2}\,\delta_{ij}\Big(\sum_{a=1}^{M+N}\mathsf{B}_{ka}(u)\mathsf{B}_{al}(v)-\sum_{a=1}^{M+N}\mathsf{B}_{ka}(v)\mathsf{B}_{al}(u)\Big)\end{aligned}\tag{5.1.7}$$

for all $1\leq i,j,k,l\leq M+N$, where the above equality may be regarded as one in the extension $\mathrm{XB}(\mathfrak{gl}_{M|N},\mathcal{G})[\![u^{\pm 1},v^{\pm 1}]\!]$, where $[\,\cdot\,,\cdot\,]$ is understood as the Lie superbracket

$$\big[\mathsf{B}_{ij}(u),\mathsf{B}_{kl}(v)\big]=\mathsf{B}_{ij}(u)\mathsf{B}_{kl}(v)-(-1)^{([i]+[j])([k]+[l])}\mathsf{B}_{kl}(v)\mathsf{B}_{ij}(u).$$

Following [MR02, Proposition 2.1], we can construct a series of central elements in $\mathrm{XB}(\mathfrak{gl}_{M|N},\mathcal{G})$ as described by the proposition below.

**Proposition 5.1.2.** *There is a series* $\mathsf{f}(u)=\mathbf{1}+\sum_{n=1}^{\infty}\mathsf{f}^{(2n)}u^{-2n}\in\mathrm{XB}(\mathfrak{gl}_{M|N},\mathcal{G})[\![u^{-2}]\!]$ *satisfying the relation*

$$\mathsf{B}(u)\mathsf{B}(-u)=\mathsf{B}(-u)\mathsf{B}(u)=\mathrm{id}\otimes\mathsf{f}(u) \tag{5.1.8}$$

*whose coefficients* $\mathsf{f}^{(2n)}$, $n\in\mathbb{Z}^{+}$, *are central elements of homogeneous* $\mathbb{Z}_2$*-degree* $\bar{0}$.

*Proof.* By multiplying the defining relations (5.1.7) with the polynomial $u^2-v^2$ and substituting $v=-u$, we yield the relation

$$\begin{aligned}\delta_{ij}\Big(\sum_{a=1}^{M+N}\mathsf{B}_{ka}(u)\mathsf{B}_{al}(-u)-\sum_{a=1}^{M+N}\mathsf{B}_{ka}(-u)\mathsf{B}_{al}(u)\Big)&\\ =(-1)^{[j][k]}2u\Big(\delta_{jk}\sum_{a=1}^{M+N}\mathsf{B}_{ia}(u)\mathsf{B}_{al}(-u)-(-1)^{[i][j]+[i][k]}\delta_{il}\sum_{a=1}^{M+N}\mathsf{B}_{ka}(-u)\mathsf{B}_{aj}(u)\Big).&\end{aligned}$$

Fixing $i=j$, evaluating at $k,l\neq i$ and $k=l=i$ in the above relation infers the equality $\mathsf{B}(u)\mathsf{B}(-u)=\mathsf{B}(-u)\mathsf{B}(u)$. Alternatively, for any $i\neq j$, evaluating at $k=j$ and $l=i$, and again at $k=j$ and $l\neq i$, in the above relation implies

$$\mathsf{B}(u)\mathsf{B}(-u)=\mathrm{id}\otimes\sum_{a=1}^{M+N}\mathsf{B}_{ka}(u)\mathsf{B}_{ak}(-u)\quad\text{for any}\quad 1\leq k\leq M+N;$$

hence, we set $\mathsf{f}(u)=\sum_{a=1}^{M+N}(-1)^{[k]+[a]}\mathsf{B}_{ka}(u)\mathsf{B}_{ak}(-u)$ for any $1\leq k\leq M+N$. Multiplying the super reflection equation (5.1.6) on the right by $\mathsf{B}_2(-v)$ yields the equation

$$R(u-v)\mathsf{B}_1(u)R(u+v)\left(\mathrm{id}^{\otimes 2}\otimes\mathsf{f}(v)\right)=\mathsf{B}_2(v)R(u+v)\mathsf{B}_1(u)R(u-v)\mathsf{B}_2(-v).$$

However, by translating $v\mapsto -v$ in the super reflection equation, we therefore have

$$\begin{aligned}\mathsf{B}_2(v)R(u+v)\mathsf{B}_1(u)R(u-v)\mathsf{B}_2(-v)&=\mathsf{B}_2(v)\mathsf{B}_2(-v)R(u-v)\mathsf{B}_1(u)R(u+v)\\&=\left(\mathrm{id}^{\otimes 2}\otimes\mathsf{f}(v)\right)R(u-v)\mathsf{B}_1(u)R(u+v),\end{aligned}$$

resulting in the relation

$$R(u-v)\mathsf{B}_1(u)R(u+v)\left(\mathrm{id}^{\otimes 2}\otimes \mathsf{f}(v)\right) = \left(\mathrm{id}^{\otimes 2}\otimes \mathsf{f}(v)\right) R(u-v)\mathsf{B}_1(u)R(u+v).$$

By the unitarity property (5.1.4) of the $R$-matrix and the fact that the coefficients of the series $\mathsf{f}(v)$ are of $\mathbb{Z}_2$-grade $\bar{0}$, we yield the desired relation

$$\mathsf{B}_1(u)\left(\mathrm{id}^{\otimes 2}\otimes \mathsf{f}(v)\right) = \left(\mathrm{id}^{\otimes 2}\otimes \mathsf{f}(v)\right)\mathsf{B}_1(u),$$

which itself implies $\mathsf{B}_{ij}(u)\mathsf{f}(v) = \mathsf{f}(v)\mathsf{B}_{ij}(u)$ for all $1 \leq i,j \leq M+N$. □

By letting $(\mathsf{f}(u)-\mathbf{1})$ denote the two-sided ideal of $\mathrm{XB}(\mathfrak{gl}_{M|N}, \mathcal{G})$ generated by the coefficients of $\mathsf{f}(u)-\mathbf{1}$, we obtain the following definition:

**Definition 5.1.3.** The *reflection superalgebra* $\mathrm{B}(\mathfrak{gl}_{M|N}, \mathcal{G})$ of $\mathfrak{gl}_{M|N}$ is the quotient of $\mathrm{XB}(\mathfrak{gl}_{M|N}, \mathcal{G})$ by the two-sided ideal $(\mathsf{f}(u)-\mathbf{1})$:

$$\mathrm{B}(\mathfrak{gl}_{M|N}, \mathcal{G}) = \mathrm{XB}(\mathfrak{gl}_{M|N}, \mathcal{G})/\left(\mathsf{f}(u)-\mathbf{1}\right).$$

Equivalently, $\mathrm{B}(\mathfrak{gl}_{M|N}, \mathcal{G})$ is the unital associative $\mathbb{C}$-superalgebra on the generators $\{B_{ij}^{(n)} \mid 1 \leq i,j \leq M+N,\ n \in \mathbb{Z}^+\}$, with $\mathbb{Z}_2$-grade $\left[B_{ij}^{(n)}\right] := [i]+[j]$ for all $n \in \mathbb{Z}^+$, subject to the *super reflection equation*

$$\begin{aligned} R(u-v)B_1(u)R(u+v)B_2(v) &= B_2(v)R(u+v)B_1(u)R(u-v) \\ \text{in} \quad &\left(\mathrm{End}\,\mathbb{C}^{M|N}\right)^{\otimes 2} \otimes \mathrm{B}(\mathfrak{gl}_{M|N}, \mathcal{G})[\![u^{\pm 1}, v^{\pm 1}]\!], \end{aligned} \tag{5.1.9}$$

where $R(u-v)$ is the $R$-matrix (5.1.1) identified with $R(u-v)\otimes \mathbf{1}$, and the unitary condition

$$B(u)B(-u) = \mathbb{1} \in \mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathrm{B}(\mathfrak{gl}_{M|N}, \mathcal{G})[\![u^{-1}]\!], \tag{5.1.10}$$

given $B(u) := \sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij}\otimes B_{ij}(u) \in \mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathrm{B}(\mathfrak{gl}_{M|N}, \mathcal{G})[\![u^{-1}]\!]$ is the matrix consisting of the series $B_{ij}(u) := \mathcal{G}_{ij}\mathbf{1} + \sum_{n=1}^{\infty} B_{ij}^{(n)}u^{-n} \in \mathrm{B}(\mathfrak{gl}_{M|N}, \mathcal{G})[\![u^{-1}]\!]$ for indices $1 \leq i,j \leq M+N$.

In terms of formal power series, the defining relations for the reflection superalgebra take the following form:

$$\begin{aligned}\big[B_{ij}(u),B_{kl}(v)\big] &= \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\big(B_{kj}(u)B_{il}(v)-B_{kj}(v)B_{il}(u)\big)\\ &\quad+\frac{(-1)^{[j][k]}}{u+v}\Big(\delta_{jk}\sum_{a=1}^{M+N}B_{ia}(u)B_{al}(v)-(-1)^{[i][j]+[i][k]}\delta_{il}\sum_{a=1}^{M+N}B_{ka}(v)B_{aj}(u)\Big)\\ &\quad-\frac{1}{u^2-v^2}\,\delta_{ij}\Big(\sum_{a=1}^{M+N}B_{ka}(u)B_{al}(v)-\sum_{a=1}^{M+N}B_{ka}(v)B_{al}(u)\Big) \qquad (5.1.11)\end{aligned}$$

and

$$\sum_{a=1}^{M+N}B_{ia}(u)B_{aj}(-u)=\delta_{ij}\mathbf{1} \qquad (5.1.12)$$

for all $1\le i,j,k,l\le M+N$.

We now establish some notation. Recalling $\mathbb{Z}^+_{M+N}=\mathbb{Z}\cap[1,M+N]$, we consider the subset $C\subset(\mathbb{Z}^+_{M+N})^2$ consisting of all pairs $(i,j)$ that satisfy any of the following inequalities:

$$\begin{gathered}1\le i,j\le p;\quad p+1\le i,j\le M;\quad M+1\le i,j\le M+q;\quad M+q+1\le i,j\le M+N;\\ 1\le i\le p,\ M+1\le j\le M+q;\quad M+1\le i\le M+q,\ 1\le j\le p;\\ p+1\le i\le M,\ M+q+1\le j\le M+N;\quad M+q+1\le i\le M+N,\ p+1\le j\le M.\end{gathered}$$

Finally, we define the subset $\mathcal{K}\subset(\mathbb{Z}^+_{M+N})^2\times\mathbb{Z}^+$ to be the following collection:

$$\mathcal{K}:=\big\{(i,j,n)\mid(i,j)\in C,\ n\in 2\mathbb{Z}^+-1\quad\text{or}\quad(i,j)\in(\mathbb{Z}^+_{M+N})^2\backslash C,\ n\in 2\mathbb{Z}^+\big\}. \qquad (5.1.13)$$

Consequently, we arrive at the following proposition.

**Proposition 5.1.4.** *The set* $\{B_{ij}^{(n)}\}_{(i,j,n)\in\mathcal{K}}$ *generates* $\mathrm{B}(\mathfrak{gl}_{M|N},\mathcal{G})$.

*Proof.* Letting $\mathcal{A}$ denote the sub-superalgebra generated by the set $\{B_{ij}^{(n)}\mid(i,j,n)\in\mathcal{K}\}$, we shall prove that $B_{ij}^{(m)}\in\mathcal{A}$ for all $1\le i,j\le M+N$ and $m\in\mathbb{Z}^+$ via induction on $m$. First, the unitary condition (5.1.12) implies

$$\sum_{r+s=m}(-1)^s\sum_{a=1}^{M+N}B_{ia}^{(r)}B_{aj}^{(s)}=0\quad\text{for}\quad m\in\mathbb{Z}^+.$$

For instance, when $m=1$ this equation becomes $(\mathcal{G}_{jj}-\mathcal{G}_{ii})B_{ij}^{(1)}=0$; hence,

$$B_{ij}^{(1)}=0 \quad \text{for all} \quad (i,j)\in(\mathbb{Z}_{M+N}^{+})^2\setminus C. \tag{5.1.14}$$

When $m=2$, we obtain $(\mathcal{G}_{ii}+\mathcal{G}_{jj})B_{ij}^{(2)}=\sum_{a=1}^{M+N}B_{ia}^{(1)}B_{aj}^{(1)}$, so (5.1.14) implies $B_{ij}^{(2)}\in\mathcal{A}$ for $(i,j)\in C$. Let us assume the induction hypothesis holds for $m-1$. By computing

$$(\mathcal{G}_{jj}+(-1)^m\mathcal{G}_{ii})B_{ij}^{(m)}=-\sum_{s=1}^{m-1}(-1)^s\sum_{a=1}^{M+N}B_{ia}^{(m-s)}B_{aj}^{(s)},$$

the induction hypothesis infers $B_{ij}^{(m)}\in\mathcal{A}$ for $(i,j)\in(\mathbb{Z}_{M+N}^{+})^2\setminus C$ if $m$ is odd and $(i,j)\in C$ if $m$ is even, concluding the proof. □

Two canonical ascending algebra filtrations on $\mathrm{B}(\mathfrak{gl}_{M|N},\mathcal{G})$ are $\mathbf{E}=\{\mathbf{E}_n\}_{n\in\mathbb{N}}$ and $\mathbf{E}'=\{\mathbf{E}'_n\}_{n\in\mathbb{N}}$ given via the respective filtration degree assignments

$$\deg_{\mathbf{E}} B_{ij}^{(n)}=n-1 \quad \text{and} \quad \deg_{\mathbf{E}'} B_{ij}^{(n)}=n.$$

for all $1\le i,j\le M+N$ and $n\in\mathbb{Z}^+$. Due to the relations (5.1.11), the associated graded superalgebra $\mathrm{gr}_{\mathbf{E}'}\mathrm{B}(\mathfrak{gl}_{M|N},\mathcal{G})=\bigoplus_{n\in\mathbb{N}}\mathbf{E}'_n/\mathbf{E}'_{n-1}$ is supercommutative.

### 5.1.2 Twisted super Yangians of type AIII

We start this subsection by first recalling the definition of the Yangian of $\mathfrak{gl}_{M|N}$ which was first introduced in §3.2.2.

**Definition 5.1.5.** The *Yangian* $\mathrm{Y}(\mathfrak{gl}_{M|N})$ of $\mathfrak{gl}_{M|N}$ is the unital associative $\mathbb{C}$-superalgebra on generators $\{T_{ij}^{(n)}\mid 1\le i,j\le M+N,\ n\in\mathbb{Z}^+\}$, equipped with $\mathbb{Z}_2$-grade $[T_{ij}^{(n)}]:=[i]+[j]$ for all $n\in\mathbb{Z}^+$, subject to the defining *RTT*-relation

$$\begin{gathered} R(u-v)T_1(u)T_2(v)=T_2(v)T_1(u)R(u-v) \\ \text{in} \quad \left(\operatorname{End}\mathbb{C}^{M|N}\right)^{\otimes 2}\otimes\mathrm{Y}(\mathfrak{gl}_{M|N})[\![u^{\pm1},v^{\pm1}]\!], \end{gathered} \tag{5.1.15}$$

where $T(u):=\sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij}\otimes T_{ij}(u)\in\operatorname{End}(\mathbb{C}^{M|N})\otimes\mathrm{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!]$ is the matrix consisting of the series $T_{ij}(u):=\delta_{ij}\mathbf{1}+\sum_{n=1}^{\infty}T_{ij}^{(n)}u^{-n}\in\mathrm{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!]$ for $1\le i,j\le M+N$, and $R(u-v)$ is the $R$-matrix (5.1.1) identified with $R(u-v)\otimes\mathbf{1}$.

On the level of power series, the $RTT$-relation (5.1.15) takes the form

$$\big[T_{ij}(u), T_{kl}(v)\big] = \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\big), \qquad (5.1.16)$$

for all $1 \leq i,j,k,l \leq M+N$, where $[\cdot\,,\cdot]$ is understood as the Lie superbracket

$$\big[T_{ij}(u), T_{kl}(v)\big] = T_{ij}(u)T_{kl}(v) - (-1)^{([i]+[j])([k]+[l])}T_{kl}(v)T_{ij}(u).$$

We note that the super Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N})$ comes equipped with a Hopf superstructure as given by the comultiplication, counit, and antipode:

$$\Delta\colon T(u) \mapsto T_{[1]}(u)T_{[2]}(u), \quad \varepsilon\colon T(u) \mapsto \mathbb{1}, \quad S\colon T(u) \mapsto T(u)^{-1}. \qquad (5.1.17)$$

Furthermore, the Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N})$ benefits from a Hopf superalgebra embedding

$$\iota\colon \mathfrak{U}(\mathfrak{gl}_{M|N}) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_{M|N}), \quad E_{ij} \mapsto (-1)^{[i]}T_{ij}^{(1)} \qquad (5.1.18)$$

and a Hopf superalgebra epimorphism

$$\mathrm{ev}\colon\ \mathrm{Y}(\mathfrak{gl}_{M|N}) \to \mathfrak{U}(\mathfrak{gl}_{M|N}), \quad T_{ij}(u) \mapsto \delta_{ij} + (-1)^{[i]}E_{ij}u^{-1} \qquad (5.1.19)$$

as can be verified by the composition $\mathrm{ev}\circ\iota = \mathrm{id}$.

Two important ascending algebra filtrations on $\mathrm{Y}(\mathfrak{gl}_{M|N})$ are $\mathbf{F} = \{\mathbf{F}_n\}_{n\in\mathbb{N}}$ and $\mathbf{F}' = \{\mathbf{F}'_n\}_{n\in\mathbb{N}}$ given via the respective filtration degree assignments

$$\deg_{\mathbf{F}} T_{ij}^{(n)} = n-1 \quad \text{and} \quad \deg_{\mathbf{F}'} T_{ij}^{(n)} = n.$$

for all $1 \leq i,j \leq M+N$ and $n \in \mathbb{Z}^+$. By the defining relations of the Yangian, the associated graded superalgebra $\mathrm{gr}_{\mathbf{F}'}\mathrm{Y}(\mathfrak{gl}_{M|N}) = \bigoplus_{n\in\mathbb{N}} \mathbf{F}'_n/\mathbf{F}'_{n-1}$ is supercommutative.

**Definition 5.1.6.** The *twisted Yangian* $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ of $\mathfrak{gl}_{M|N}$ is the sub-superalgebra of $\mathrm{Y}(\mathfrak{gl}_{M|N})$ generated by the coefficients $\big\{S_{ij}^{(n)} \mid 1 \leq i,j \leq M+N,\ n \in \mathbb{Z}^+\big\}$ of

$$S(u) := T(u)\mathcal{G}T(-u)^{-1} \in \big(\mathrm{End}\,\mathbb{C}^{M|N}\big)^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!], \qquad (5.1.20)$$

where $S(u) = \sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]}E_{ij} \otimes S_{ij}(u) \in \mathrm{End}(\mathbb{C}^{M|N}) \otimes \mathrm{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!]$ is the matrix consisting of the series $S_{ij}(u) := \mathcal{G}_{ij}\mathbf{1} + \sum_{n=1}^{\infty} S_{ij}^{(n)}u^{-n} \in \mathrm{Y}(\mathfrak{gl}_{M|N})[\![u^{-1}]\!]$.

By writing the matrix $T(u)^{-1} = \sum_{i,j=1}^{M+N}(-1)^{[i][j]+[j]} E_{ij} \otimes T^{\bullet}_{ij}(u)$ where $T^{\bullet}_{ij}(u)$ is written as $\delta_{ij}\mathbf{1} + \sum_{n=1}^{\infty} T^{\bullet(n)}_{ij} u^{-n}$, then the series $S_{ij}(u)$ takes the form

$$S_{ij}(u) = \sum_{a=1}^{M+N} \mathcal{G}_{aa} T_{ia}(u) T^{\bullet}_{aj}(-u). \tag{5.1.21}$$

We note that one can compute the coefficients of $T^{\bullet}_{ij}(u)$ explicitly as

$$T^{\bullet(n)}_{ij} = -T^{(n)}_{ij} + \sum_{s=2}^{n}(-1)^s \sum_{\sum_{j=1}^{s} k_j = n} \left( \sum_{a_1,a_2,\ldots,a_{s-1}=1}^{M+N} T^{(k_1)}_{ia_1} T^{(k_2)}_{a_1a_2} \cdots T^{(k_s)}_{a_{s-1}j} \right),$$

where $k_j \in \mathbb{Z}^+$ for each $k_j$ in the sum $\sum_{j=1}^{s} k_j = n$. In particular, we can conclude $S_{ij}(u)$ is homogeneous of degree $[S_{ij}(u)] = [i]+[j]$ since $[T^{\bullet}_{aj}(-u)] = [a]+[j]$.

As a sub-superalgebra of the Hopf superalgebra $\mathrm{Y}(\mathfrak{gl}_{M|N})$, a natural inquiry is whether or not the Hopf superstructure on the Yangian restricts to one on the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$. The answer to this question is in the negative, as the twisted super Yangian appears instead as a left coideal of $\mathrm{Y}(\mathfrak{gl}_{M|N})$.

**Proposition 5.1.7.** *The twisted Yangian* $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ *is a left coideal of* $\mathrm{Y}(\mathfrak{gl}_{M|N})$:

$$\Delta(\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}) \subseteq \mathrm{Y}(\mathfrak{gl}_{M|N}) \otimes \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}.$$

*Proof.* Having the comultiplication map $\Delta$ on $\mathrm{Y}(\mathfrak{gl}_{M|N})$ act on $T(u)T(u)^{-1} = \mathbb{1}$ infers the equality

$$\Delta(T(u)^{-1}) = T_{[2]}(u)^{-1} T_{[1]}(u)^{-1}.$$

Thus, $\Delta(T^{\bullet}_{ij}(u)) = \sum_{k=1}^{M+N}(-1)^{([i]+[k])([k]+[j])} T^{\bullet}_{kj}(u) \otimes T^{\bullet}_{ik}(u)$. Hence, for any indices $1 \le i, j \le M+N$, one can compute

$$\begin{aligned}
\Delta(S_{ij}(u)) &= \sum_{c=1}^{M+N} \mathcal{G}_{cc} \Delta(T_{ic}(u)) \Delta(T^{\bullet}_{cj}(-u)) \\
&= \sum_{a,b,c=1}^{M+N} (-1)^{([c]+[b])([b]+[j])} \mathcal{G}_{cc} (T_{ia}(u) \otimes T_{ac}(u))(T^{\bullet}_{bj}(-u) \otimes T^{\bullet}_{cb}(-u)) \\
&= \sum_{a,b=1}^{M+N} (-1)^{([a]+[b])([b]+[j])} T_{ia}(u) T^{\bullet}_{bj}(-u) \otimes S_{ab}(u),
\end{aligned}$$

completing the proof. □

Note that the filtrations $\mathbf{F}$ and $\mathbf{F}'$ on $\mathrm{Y}(\mathfrak{gl}_{M|N})$ endow filtrations on the twisted super Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$, which we will also denote $\mathbf{F}$ and $\mathbf{F}'$ respectively. Such filtrations are given by the degree assignments

$$\deg_{\mathbf{F}} S_{ij}^{(n)} = n-1 \quad \text{and} \quad \deg_{\mathbf{F}'} S_{ij}^{(n)} = n.$$

for all $1 \le i,j \le M+N$ and $n \in \mathbb{Z}^+$.

**Lemma 5.1.8.** *Let $\overline{S}_{ij}^{(n)}$ denote the image of $S_{ij}^{(n)}$ in $n$-th graded component $\mathbf{F}'_n/\mathbf{F}'_{n-1}$ of the graded superalgebra $\mathrm{gr}_{\mathbf{F}'}\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw} = \bigoplus_{n\in\mathbb{N}} \mathbf{F}'_n/\mathbf{F}'_{n-1}$. Fixing a total order '$\preceq$' on the index set $\mathcal{K}$ defined by (5.1.13), then the collection of all ordered monomials of the form*

$$\overline{S}_{i_1j_1}^{(n_1)}\overline{S}_{i_2j_2}^{(n_2)}\cdots\overline{S}_{i_kj_k}^{(n_k)} \tag{5.1.22}$$

*with $(i_a,j_a,n_a) \in \mathcal{K}$, $(i_a,j_a,n_a) \preceq (i_{a+1},j_{a+1},n_{a+1})$, and $(i_a,j_a,n_a) \neq (i_{a+1},j_{a+1},n_{a+1})$ if $[i_a]+[j_a] = \bar{1}$, are linearly independent.*

*Proof.* We recall the associated graded superalgebra $\mathrm{gr}_{\mathbf{F}'}\mathrm{Y}(\mathfrak{gl}_{M|N})$ and let us have $\overline{T}_{ij}^{(n)}$ denote the image of the generator $T_{ij}^{(n)}$ in its $n$-th graded component. As was shown in the proof of [Gow07, Theorem 1], if we endow a total order $\preceq$ on the set $\mathcal{I} = (\mathbb{Z}^+_{M+N})^2 \times \mathbb{Z}^+$, then the collection of all ordered monomials of the form

$$\overline{T}_{i_1j_1}^{(n_1)}\overline{T}_{i_2j_2}^{(n_2)}\cdots\overline{T}_{i_kj_k}^{(n_k)} \tag{5.1.23}$$

with $(i_a,j_a,n_a) \in \mathcal{I}$, $(i_a,j_a,n_a) \preceq (i_{a+1},j_{a+1},n_{a+1})$, and $(i_a,j_a,n_a) \neq (i_{a+1},j_{a+1},n_{a+1})$ if $[i_a]+[j_a] = \bar{1}$, forms a basis for $\mathrm{gr}_{\mathbf{F}'}\mathrm{Y}(\mathfrak{gl}_{M|N})$.

Let us now introduce another filtration $\overline{\mathbf{F}} = \{\overline{\mathbf{F}}_n\}_{n\in\mathbb{N}}$ but instead on $\mathrm{gr}_{\mathbf{F}'}\mathrm{Y}(\mathfrak{gl}_{M|N})$ via $\deg_{\overline{\mathbf{F}}} \overline{T}_{ij}^{(n)} = n-1$. Therefore, from the description (5.1.21), we have

$$\overline{S}_{ij}^{(n)} \equiv (\mathcal{G}_{jj} + (-1)^{n+1}\mathcal{G}_{ii})\overline{T}_{ij}^{(n)} \mod \overline{\mathbf{F}}_{n-2}, \tag{5.1.24}$$

so $\bar{S}_{ij}^{(n)} \equiv \pm 2\overline{T}_{ij}^{(n)} \bmod \overline{\mathbf{F}}_{n-2}$ if and only if $(i,j,k) \in \mathcal{K}$.

If we assume to the contrary that there exists a non-trivial linear combination $A$ of ordered monomials of the form (5.1.22) such that $A = 0$, then let $R$ denote the linear combination of those monomials occurring in $A$ of maximal $\overline{\mathbf{F}}$-filtration degree say $\alpha$.

However, the equivalence (5.1.24) would imply $0 = A \equiv R \equiv \widetilde{R} \bmod \overline{\mathbf{F}}_{\alpha-1}$, where $\widetilde{R}$ is a non-trivial linear combination of ordered monomials of the form (5.1.23) of $\overline{\mathbf{F}}$-filtration degree $\alpha$, but this contradicts their linear independence. □

We can now establish the following isomorphism:

**Theorem 5.1.9.** *There is a superalgebra isomorphism*

$$\varphi\colon \mathrm{B}(\mathfrak{gl}_{M|N},\mathcal{G}) \xrightarrow{\sim} \mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}, \quad B(u) \mapsto S(u). \tag{5.1.25}$$

*Proof.* It is immediate that the map $\varphi$ is homogeneous and surjective. To show $\varphi$ is a superalgebra morphism, one can first readily check $S(u)S(-u) = \mathbb{1}$ is satisfied since $\mathcal{G}^2 = \mathrm{id}$. To show to $S(u)$ satisfies the super reflection equation, we require the use of the following equations obtained from the $RTT$-relation (5.1.15):

$$T_1(-u)^{-1}R(u+v)T_2(v) = T_2(v)R(u+v)T_1(-u)^{-1}, \tag{5.1.26}$$

$$R(u-v)T_1(-u)^{-1}T_2(-v)^{-1} = T_2(-v)^{-1}T_1(-u)^{-1}R(u-v), \tag{5.1.27}$$

$$T_1(u)R(u+v)T_2(-v)^{-1} = T_2(-v)^{-1}R(u+v)T_1(u). \tag{5.1.28}$$

By using (5.1.26) and the fact that $\mathcal{G}_i$ commutes with $T_j(u)$ and $T_j(-u)^{-1}$ for integers $1 \le i \ne j \le 2$, the expression $R(u-v)S_1(u)R(u+v)S_2(v)$ is given by

$$\begin{aligned} &R(u-v)T_1(u)\mathcal{G}_1T_1(-u)^{-1}R(u+v)T_2(v)\mathcal{G}_2T_2(-v)^{-1} \\ &\qquad = R(u-v)T_1(u)\mathcal{G}_1T_2(v)R(u+v)T_1(-u)^{-1}\mathcal{G}_2T_2(-v)^{-1} \\ &\qquad = R(u-v)T_1(u)T_2(v)\mathcal{G}_1R(u+v)\mathcal{G}_2T_1(-u)^{-1}T_2(-v)^{-1} \\ &\qquad = T_2(v)T_1(u)R(u-v)\mathcal{G}_1R(u+v)\mathcal{G}_2T_1(-u)^{-1}T_2(-v)^{-1}, \end{aligned}$$

where we used the $RTT$-relation in the last equality. Furthermore, since $\mathcal{G}_1\mathcal{G}_2 = \mathcal{G}_2\mathcal{G}_1$ and $\mathcal{G}_1P = P\mathcal{G}_2$, then $R(u-v)\mathcal{G}_1R(u+v)\mathcal{G}_2 = \mathcal{G}_2R(u-v)\mathcal{G}_1R(u+v)$. Therefore, by using this equality and equations (5.1.27), (5.1.28), the above expression becomes

$$\begin{aligned} &T_2(v)T_1(u)\mathcal{G}_2R(u+v)\mathcal{G}_1R(u-v)T_1(-u)^{-1}T_2(-v)^{-1} \\ &\qquad = T_2(v)T_1(u)\mathcal{G}_2R(u+v)\mathcal{G}_1T_2(-v)^{-1}T_1(-u)^{-1}R(u-v) \\ &\qquad = T_2(v)\mathcal{G}_2T_1(u)R(u+v)T_2(-v)^{-1}\mathcal{G}_1T_1(-u)^{-1}R(u-v) \\ &\qquad = T_2(v)\mathcal{G}_2T_2(-v)^{-1}R(u+v)T_1(u)\mathcal{G}_1T_1(-u)^{-1}R(u-v), \end{aligned}$$

which is the expression $S_2(v)R(u+v)S_1(u)R(u-v)$.

To show the morphism $\varphi$ is injective, it suffices to prove that the associated morphism

$$\operatorname{gr}\varphi\colon\ \operatorname{gr}_{\mathbf{E}'}\mathrm{B}(\mathfrak{gl}_{M|N},\mathcal{G})\to\operatorname{gr}_{\mathbf{F}'}\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$$

is so. However, by Proposition (5.1.4), we know the set $\{\overline{B}_{ij}^{(n)}\}_{(i,j,n)\in\mathcal{K}}$ generates $\operatorname{gr}_{\mathbf{E}'}\mathrm{B}(\mathfrak{gl}_{M|N},\mathcal{G})$, so the map $\operatorname{gr}\varphi$ is injective by Lemma (5.1.8). □

Through the course of the proof for Theorem 5.1.9, we also established an explicit basis for the reflection superalgebra and twisted super Yangian. Such is described by the corollary below.

**Corollary 5.1.10** (PBW Theorem). *Fix a total order '$\preceq$' on the index set $\mathcal{K}$ defined by* (5.1.13).

(i) *The collection of all ordered monomials of the form*

$$B_{i_1j_1}^{(n_1)}B_{i_2j_2}^{(n_2)}\cdots B_{i_kj_k}^{(n_k)},\quad \textit{where}\quad (i_a,j_a,n_a)\in\mathcal{K}\quad \textit{for}\quad 1\le a\le k, \tag{5.1.29}$$

*such that* $(i_a,j_a,n_a)\preceq(i_{a+1},j_{a+1},n_{a+1})$, *and* $(i_a,j_a,n_a)\neq(i_{a+1},j_{a+1},n_{a+1})$ *if* $[i_a]+[j_a]=\bar{1}$, *constitutes a basis for* $\mathrm{B}(\mathfrak{gl}_{M|N},\mathcal{G})$.

(ii) *The collection of all ordered monomials of the form*

$$S_{i_1j_1}^{(n_1)}S_{i_2j_2}^{(n_2)}\cdots S_{i_kj_k}^{(n_k)},\quad \textit{where}\quad (i_a,j_a,n_a)\in\mathcal{K}\quad \textit{for}\quad 1\le a\le k, \tag{5.1.30}$$

*such that* $(i_a,j_a,n_a)\preceq(i_{a+1},j_{a+1},n_{a+1})$, *and* $(i_a,j_a,n_a)\neq(i_{a+1},j_{a+1},n_{a+1})$ *if* $[i_a]+[j_a]=\bar{1}$, *constitutes a basis for* $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$.

As a benefit of Theorem 5.1.9, we can also express the twisted super Yangian in terms of generators and relations, which will be particularly useful in investigating its representation theory in the subsequent section.

**Corollary 5.1.11.** *The twisted Yangian* $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ *is the unital associative* $\mathbb{C}$*-superalgebra on generators* $\{S_{ij}^{(n)} \mid 1 \le i,j \le M+N,\ n \in \mathbb{Z}^+\}$*, equipped with* $\mathbb{Z}_2$*-grade* $[S_{ij}^{(n)}] := [i]+[j]$ *for all* $n \in \mathbb{Z}^+$*, subject to the defining relations*

$$\begin{aligned}\big[S_{ij}(u), S_{kl}(v)\big] &= \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\big(S_{kj}(u)S_{il}(v) - S_{kj}(v)S_{il}(u)\big) \\ &\quad + \frac{(-1)^{[j][k]}}{u+v}\Big(\delta_{jk}\sum_{a=1}^{M+N} S_{ia}(u)S_{al}(v) - (-1)^{[i][j]+[i][k]}\delta_{il}\sum_{a=1}^{M+N} S_{ka}(v)S_{aj}(u)\Big) \\ &\quad - \frac{1}{u^2-v^2}\,\delta_{ij}\Big(\sum_{a=1}^{M+N} S_{ka}(u)S_{al}(v) - \sum_{a=1}^{M+N} S_{ka}(v)S_{al}(u)\Big) \qquad (5.1.31)\end{aligned}$$

*and*

$$\sum_{a=1}^{M+N} S_{ia}(u)S_{aj}(-u) = \delta_{ij}\mathbf{1} \qquad (5.1.32)$$

*for all* $1 \le i,j,k,l \le M+N$.

We end this section by providing justification to the terminology 'twisted' used in name of the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$. Namely, by defining $\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ as the Rees superalgebra of $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$, we will establish a superalgebra isomorphism $\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}/\hbar\,\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw} \cong \mathfrak{U}(\mathfrak{gl}_{M|N}[z]^\vartheta)$, where $\mathfrak{gl}_{M|N}[z]^\vartheta$ is the fixed-point Lie sub-superalgebra of $\mathfrak{gl}_{M|N}[z]$ under some involutive automorphism $\vartheta$, called the *twisted current Lie superalgebra.* To this end, consider the involution $\vartheta \in \mathrm{Aut}(\mathfrak{gl}_{M|N})$ given by

$$\vartheta\colon \mathfrak{gl}_{M|N} \to \mathfrak{gl}_{M|N}, \quad E_{ij} \mapsto \mathcal{G}_{ii}\mathcal{G}_{jj}E_{ij}. \qquad (5.1.33)$$

We find that the fixed-point Lie sub-superalgebra $\mathfrak{gl}_{M|N}^\vartheta$ of $\mathfrak{gl}_{M|N}$ under the involutive automorphism $\vartheta$ is generated by the operators

$$E_{ij} + \vartheta(E_{ij}) = (1+\mathcal{G}_{ii}\mathcal{G}_{jj})E_{ij} \in \mathfrak{gl}_{M|N}^\vartheta \quad \text{for all} \quad 1 \le i,j \le M+N.$$

In particular, by setting $p' = M-p$ and $q' = N-q$, one can verify the Lie superalgebra isomorphism

$$\mathfrak{gl}_{M|N}^\vartheta \cong \mathfrak{gl}_{p|q} \oplus \mathfrak{gl}_{p'|q'}$$

Extending $\vartheta$ to an involutive automorphism of $\mathfrak{gl}_{M|N}[z]$ via the assignment

$$\vartheta(f(z)) = \vartheta(f)(-z) \quad \text{for all} \quad f(z) \in \mathfrak{gl}_{M|N}[z],$$

define the *twisted current Lie superalgebra* $\mathfrak{gl}_{M|N}[z]^{\vartheta}$ as the fixed-point sub-superalgebra of $\mathfrak{gl}_{M|N}[z]$ under the involutive automorphism $\vartheta$:

$$\mathfrak{gl}_{M|N}[z]^{\vartheta} := \big\{ g(z) \in \mathfrak{gl}_{M|N}[z] \mid \vartheta(g(z)) = g(z) \big\}.$$

In particular, we find $\mathfrak{gl}_{M|N}[z]^{\vartheta}$ is generated by the operators

$$E_{ij}^{(n)}(z) := E_{ij}z^n + \vartheta(E_{ij})(-z)^n = (1+(-1)^n \mathcal{G}_{ii}\mathcal{G}_{jj})E_{ij}z^n \in \mathfrak{gl}_{M|N}[z]^{\vartheta} \tag{5.1.34}$$

with $1 \leq i,j \leq M+N$, $n \in \mathbb{N}$, subject only to the relations

$$\big[E_{ij}^{(m)}(z), E_{kl}^{(n)}(z)\big] = (1+(-1)^m \mathcal{G}_{ii}\mathcal{G}_{jj})\big(\delta_{jk}E_{il}^{(m+n)}(z) - \delta_{il}(-1)^{([i]+[j])([k]+[l])}E_{kj}^{(m+n)}(z)\big)$$

and

$$(1-(-1)^n \mathcal{G}_{ii}\mathcal{G}_{jj})E_{ij}^{(n)}(z) = 0.$$

Now consider the following corollary to the PBW Theorem:

**Corollary 5.1.12.** *There is an* $\mathbb{N}$*-graded superalgebra isomorphism*

$$\Psi\colon \mathfrak{U}(\mathfrak{gl}_{M|N}[z]^{\vartheta}) \xrightarrow{\sim} \mathrm{gr}_{\mathbf{F}}\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}, \quad E_{ij}^{(n-1)}(z) \mapsto (-1)^{[i]}\mathcal{G}_{jj}\overline{S}_{ij}^{(n)} \tag{5.1.35}$$

*for* $1 \leq i,j \leq M+N$ *and* $n \in \mathbb{Z}^+$.

*Proof.* To show $\Psi\colon \mathfrak{gl}_{M|N}[z]^{\vartheta} \xrightarrow{\sim} \mathrm{Lie}(\mathrm{gr}_{\mathbf{F}}\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw})$ is an $\mathbb{N}$-graded Lie superalgebra morphism, one passes the relations (5.1.31) and (5.1.32) to the associated graded superalgebra to yield the respective relations

$$\begin{aligned}\big[\overline{S}_{ij}^{(m)}, \overline{S}_{kl}^{(n)}\big] = \delta_{jk}(-1)^{[k]}&(\mathcal{G}_{jj} + (-1)^{m-1}\mathcal{G}_{ii})\overline{S}_{il}^{(m+n-1)} \\ &- \delta_{il}(-1)^{[i][j]+[i][k]+[j][k]}(\mathcal{G}_{ii} + (-1)^{m-1}\mathcal{G}_{jj})\overline{S}_{kj}^{(m+n-1)}\end{aligned}$$

and

$$(\mathcal{G}_{ii} + (-1)^n\mathcal{G}_{jj})\overline{S}_{ij}^{(n)} = 0$$

for all $1 \le i,j,k,l \le M+N$ and $m,n \in \mathbb{Z}^+$. Hence, the desired relations follow from multiplying the first equation above by $(-1)^{[i]+[k]}\mathcal{G}_{jj}\mathcal{G}_{ll}$ and the second by $(-1)^{[i]+n}\mathcal{G}_{jj}$.

Thus, $\Psi$ extends to a morphism of superalgebras $\mathfrak{U}(\mathfrak{gl}_{M|N}[z]^{\vartheta}) \to \mathrm{gr}_{\mathbf{F}}\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ which is also $\mathbb{N}$-graded. The injectivity of $\Psi$ follows from Corollary 5.1.10, whilst $\Psi$ is surjective since $\mathrm{gr}_{\mathbf{F}}\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ is generated by the elements $\overline{S}_{ij}^{(n)}$ for $1 \le i,j \le M+N$ and $n \in \mathbb{Z}^+$. □

**Definition 5.1.13.** Given the tensor product $\mathbb{C}[\hbar] \otimes \mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw} = \mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}[\hbar]$ where $\hbar$ is a formal element of $\mathbb{Z}_2$-degree $\bar{0}$, the *twisted Yangian* $\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ is defined as the Rees superalgebra of $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ with respect to the filtration $\mathbf{F}$:

$$\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw} := R_\hbar(\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}) = \bigoplus_{n\in\mathbb{N}} \hbar^n \mathbf{F}_n \subset \mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}[\hbar].$$

We recall that given any superalgebra $\mathcal{A}$ over $\mathbb{C}$, a *flat deformation of* $\mathcal{A}$ (over $\mathbb{C}[\hbar]$) is a superalgebra $\mathcal{A}_\hbar$ over $\mathbb{C}[\hbar]$ such that:

(i) $\mathcal{A}_\hbar$ is flat as a $\mathbb{C}[\hbar]$-module.

(ii) The quotient $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ is isomorphic to $\mathcal{A}$ as a superalgebra.

Regarding $\mathbb{C}[\hbar] = \bigoplus_{k\in\mathbb{N}} \hbar^k\mathbb{C}$ as an $\mathbb{N}$-graded ring, such deformation is called *homogeneous* if both $\mathcal{A}$ and $\mathcal{A}_\hbar$ are $\mathbb{N}$-graded modules where the isomorphism $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \cong \mathcal{A}$ is grade-preserving.

**Proposition 5.1.14.** *The twisted Yangian* $\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ *is a homogeneous flat deformation of* $\mathfrak{U}(\mathfrak{gl}_{M|N}[z]^{\vartheta})$. *Furthermore, there is a superalgebra isomorphism*

$$\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}/(\hbar-\lambda)\,\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw} \cong \mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw} \quad \textit{for all} \quad \lambda \in \mathbb{C}^*.$$

*Proof.* We observe that $\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ is flat since it is torsion-free as a $\mathbb{C}[\hbar]$-subalgebra of $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}[\hbar]$. In particular, the composition of the superalgebra isomorphism

$$\begin{aligned} \phi\colon\ & \mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}/\hbar\,\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw} \xrightarrow{\sim} \mathrm{gr}_{\mathbf{F}}\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw} \\ & \hbar^{n-1}S_{ij}^{(n)} \bmod \hbar\,\mathrm{Y}_\hbar(\mathfrak{gl}_{M|N},\mathcal{G})^{tw} \mapsto \overline{S}_{ij}^{(n)} \end{aligned}$$

for $1 \leq i, j \leq M+N$, $n \in \mathbb{Z}^{+}$, with the inverse of the isomorphism $\Psi$ (5.1.35) gives the desired $\mathbb{N}$-graded Hopf superalgebra isomorphism

$$\Psi^{-1} \circ \phi\colon \ \mathrm{Y}_{\hbar}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw} / \hbar\, \mathrm{Y}_{\hbar}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw} \xrightarrow{\sim} \mathfrak{U}(\mathfrak{gl}_{M|N}[z]^{\vartheta}).$$

For the second claim, consider the morphism $\mathrm{ev}_{\lambda}\colon \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}[\hbar] \to \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ induced by the assignment $\hbar \mapsto \lambda$. The restriction $\mathrm{ev}_{\lambda}^{R}$ of $\mathrm{ev}_{\lambda}$ to $R_{\hbar}(\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw})$ will still remain surjective and its kernel is given by

$$\ker(\mathrm{ev}_{\lambda}^{R}) = R_{\hbar}(\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}) \cap (\hbar - \lambda)\, \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}[\hbar] = (\hbar - \lambda) R_{\hbar}(\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}),$$

proving the proposition. □

## 5.2 Representation Theory of Twisted Super Yangians

### 5.2.1 Highest weight theory

From equation (5.1.21) we have $\mathcal{G}_{kk} S_{kk}^{(1)} = 2T_{kk}^{(1)}$, so by identifying the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_{M|N}$ with its image in $\mathrm{Y}(\mathfrak{gl}_{M|N})$ under the embedding (5.1.18), we deduce

$$\mathfrak{h} = \bigoplus_{k=1}^{M+N} \mathbb{C} S_{kk} \subset \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}.$$

Declaring the basis $\{\mathcal{H}_k\}_{k=1}^{M+N}$ of $\mathfrak{h}$ defined by $\mathcal{H}_k := \frac{1}{2}(-1)^{[k]} \mathcal{G}_{kk} S_{kk}^{(1)}$, we consider its dual basis $\{\varepsilon_k\}_{k=1}^{M+N} \subset \mathfrak{h}^*$ to yield the root system

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq M+N,\ i \neq j\}$$

of $\mathfrak{gl}_{M|N}$. We use the standard root system decomposition $\Phi = \Phi^{+} \sqcup \Phi^{-}$, where $\Phi^{+} = \{\varepsilon_i - \varepsilon_j \in \Phi \mid i < j\}$ and $\Phi^{-} = \{\varepsilon_i - \varepsilon_j \in \Phi \mid i > j\}$ and call the linear functionals in $\Phi^{\pm}$ as positive/negative roots. Via the relations (5.1.31), we compute

$$\big[S_{kl}^{(1)}, S_{ij}(u)\big] = (-1)^{[i][k]+[i][l]+[k][l]} (\mathcal{G}_{kk} + \mathcal{G}_{ll}) \big(\delta_{il} S_{kj}(u) - \delta_{jk} S_{il}(u)\big), \tag{5.2.1}$$

and hence

$$\big[\mathcal{H}_k, S_{ij}(u)\big] = (\delta_{ik} - \delta_{jk}) S_{ij}(u) = (\varepsilon_i - \varepsilon_j)(\mathcal{H}_k) S_{ij}(u) \tag{5.2.2}$$

for all $1 \le i, j, k \le M+N$. In particular, we have the following decomposition in terms of the root lattice $\mathbb{Z}\Phi$:

$$\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw} = \bigoplus_{\alpha \in \mathbb{Z}\Phi} \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}_{\alpha},$$

where $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}_{\alpha} = \{X \in \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$. We also have familiar notions of weights and weight vectors for representations $V$ of $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$: for any functional $\lambda \in \mathfrak{h}^*$, provided

$$V_\lambda := \{v \in V \mid H \cdot v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\} \neq 0,$$

then $\lambda$ is called a *weight*, $V_\lambda$ is called a *weight space*, and nonzero vectors in $V_\lambda$ are called *weight vectors*. We endow a partial ordering '$\preccurlyeq$' on the set of weights of $V$ via the rule $\omega \preccurlyeq \mu \Leftrightarrow \mu - \omega$ is an $\mathbb{N}$-linear combination of positive roots of $\mathfrak{gl}_{M|N}$. Furthermore, since $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}_{\alpha}(V_\lambda) \subseteq V_{\lambda+\alpha}$, then

$$\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}_{\alpha}\Big(\bigoplus\nolimits_{\mu \in \mathfrak{h}^*} V_\mu\Big) \subseteq \bigoplus\nolimits_{\mu \in \mathfrak{h}^*} V_\mu. \tag{5.2.3}$$

**Definition 5.2.1.** A representation $V$ of the twisted super Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ is defined as a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}\xi = V$, and

$$\begin{aligned} S_{ij}(u)\xi &= 0 \qquad \text{for all} \quad 1 \le i < j \le M+N \\ \text{and} \quad S_{kk}(u)\xi &= \lambda_k(u)\xi \qquad \text{for all} \quad 1 \le k \le M+N, \end{aligned} \tag{5.2.4}$$

where $\lambda_k(u)$ is some formal series

$$\lambda_k(u) = \mathcal{G}_{kk} + \sum_{n=1}^{\infty} \lambda_k^{(n)} u^{-n} \in \mathbb{C}[\![u^{-1}]\!]. \tag{5.2.5}$$

We say that $\xi$ is the *highest weight vector* of $V$ and call the tuple $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ of formal series the *highest weight* of $V$.

To prove the first theorem of this section, we will need the following technical lemma. We note that many of the techniques used in the proof below arise from those used in the proof of [MR02, Theorem 4.1].

**Lemma 5.2.2.** *Let $\mathcal{J}$ be the left graded ideal of* $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ *generated by the coefficients of* $S_{ij}(u)$ *for all* $1 \le i < j \le M+N$. *Then:*

(i) $S_{ij}(u)S_{kk}(v) \equiv 0 \bmod \mathcal{J}$ *for all* $1 \le i < j \le M+N$ *and* $1 \le k \le M+N$,

(ii) $\big[S_{kk}(u), S_{ll}(v)\big] \equiv 0 \bmod \mathcal{J}$ *for all* $1 \le k, l \le M+N$.

*Proof.* For brevity, we shall use '$\equiv$' to denote the equivalence of elements in $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ modulo $\mathcal{J}$.

(i) We shall prove the statement via reverse strong induction on $1 \le k \le M+N$. For the base case $k = M+N$, the relations (5.1.31) imply $S_{ij}(u)S_{M+N,M+N}(v) \equiv 0$ for indices $1 \le i < j \le M+N-1$. When $1 \le i < j = M+N$, the same relations imply

$$S_{i,M+N}(u)S_{M+N,M+N}(v) \equiv -\frac{1}{u+v}S_{i,M+N}(u)S_{M+N,M+N}(v),$$

so $S_{i,M+N}(u)S_{M+N,M+N}(v) \equiv 0$ as well.

Suppose now the hypothesis holds down to $k+1$. We make the initial observation that for any indices $j, l$ such that $1 \le i < j, l$, then the relations (5.1.31) imply

$$S_{ij}(u)S_{jl}(v) \equiv \frac{1}{u+v}(-1)^{[j]} \sum_{a=l}^{M+N} S_{ia}(u)S_{al}(v). \tag{5.2.6}$$

Since the right side of the equivalence (5.2.6) is independent of the index $j$, we have the following equivalence for all indices $i, j_1, j_2, l$ such that $1 \le i < j_1, j_2, l$:

$$S_{ij_1}(u)S_{j_1 l}(v) \equiv (-1)^{[j_1]+[j_2]} S_{ij_2}(u)S_{j_2 l}(v). \tag{5.2.7}$$

We demarcate the remaining inductive proof of (i) into three steps addressing the respective cases $i < k$, $i = k$, and $i > k$.

*Step 1.* First assuming $i < k$ such that $j \ne k$, the relations (5.1.31) immediately show $S_{ij}(u)S_{kk}(v) \equiv 0$, so we may suppose $1 \le i < j = k$ without loss of generality. By the equivalences (5.2.6) and (5.2.7), we obtain

$$S_{ik}(u)S_{kk}(v) \equiv \frac{1}{u+v}(-1)^{[k]} \sum_{a=k}^{M+N} S_{ia}(u)S_{ak}(v) \equiv \frac{(-1)^{[k]}(M+1-k)-N}{u+v} S_{ik}(u)S_{kk}(v),$$

since $\sum_{a=k}^{M+N}(-1)^{[a]} = (-1)^{[k]}(M+1-k)-N$. Thus, $S_{ik}(u)S_{kk}(v) \equiv 0$.

*Step 2.* When $1 \leq i = k < j$, the relations (5.1.31) give

$$\begin{aligned} S_{kj}(u)S_{kk}(v) \equiv \frac{1}{u-v}(-1)^{[k]}\big(S_{kj}(u)S_{kk}(v) - S_{kj}(v)S_{kk}(u)\big) \\ - \frac{1}{u+v}(-1)^{[k]}\sum_{a=j}^{M+N} S_{ka}(v)S_{aj}(u). \end{aligned}$$

However, by the formula (5.2.7) the sum $\sum_{a=j}^{M+N} S_{ka}(v)S_{aj}(u)$ is equivalent to the series $(M+1-j-(-1)^{[j]}N)S_{kj}(u)S_{jj}(v)$, which itself is equivalent to zero by induction hypothesis. We therefore have the relation

$$\frac{u-v-(-1)^{[k]}}{u-v}S_{kj}(u)S_{kk}(v) + \frac{(-1)^{[k]}}{u-v}S_{kj}(v)S_{kk}(u) \equiv 0. \tag{5.2.8}$$

Furthermore, by exchanging $u \leftrightarrow v$ in (5.2.8), we obtain

$$-\frac{(-1)^{[k]}}{u-v}S_{kj}(u)S_{kk}(v) + \frac{u-v+(-1)^{[k]}}{u-v}S_{kj}(v)S_{kk}(u) \equiv 0. \tag{5.2.9}$$

Hence, taking the difference of (5.2.8) and (5.2.9), infers $S_{kj}(u)S_{kk}(v) \equiv S_{kj}(v)S_{kk}(u)$, so either (5.2.8) or (5.2.9) will establish $S_{kj}(u)S_{kk}(v) \equiv 0$.

*Step 3.* Lastly, when $1 \leq k < i < j$, the relations (5.1.31) imply both

$$S_{ij}(u)S_{kk}(v) \equiv \frac{1}{u-v}(-1)^{[i][j]}\big(S_{kj}(u)S_{ik}(v) - S_{kj}(v)S_{ik}(v)\big)$$

and

$$\begin{aligned} S_{kj}(u)S_{ik}(v) \equiv \frac{1}{u-v}(-1)^{[i][j]+[i][k]+[j][k]}\big(S_{ij}(u)S_{kk}(v) - S_{ij}(v)S_{kk}(u)\big) \\ - \frac{1}{u+v}(-1)^{[i][j]+[i][k]+[j][k]}\sum_{a=j}^{M+N} S_{ia}(v)S_{aj}(u). \end{aligned}$$

By the equivalence (5.2.7), the sum $\sum_{a=j}^{M+N} S_{ia}(v)S_{aj}(u)$ is equivalent to the series $(M+1-j-(-1)^{[j]}N)S_{ij}(v)S_{jj}(u)$, which itself is equivalent to zero by induction hypothesis. Therefore, $S_{kj}(u)S_{ik}(v) \equiv S_{kj}(v)S_{ik}(u)$ and so $S_{ij}(u)S_{kk}(v) \equiv 0$.

(ii) We shall first prove that $[S_{kk}(u), S_{kk}(v)] \equiv 0$ for all $1 \leq k \leq M+N$ via reverse strong induction on $k$. To this end, it will be useful to define the element

$$\beta_k(u,v) := \sum\nolimits_{a=k+1}^{M+N} \big(S_{ka}(u)S_{ak}(v) - S_{ka}(v)S_{ak}(u)\big). \tag{5.2.10}$$

In particular, the relations (5.1.31) imply

$$\begin{aligned}\big[S_{kk}(u), S_{kk}(v)\big] \equiv \frac{(-1)^{[k]}}{u-v}\big[S_{kk}(u), S_{kk}(v)\big] + \frac{(-1)^{[k]}}{u+v}\big[S_{kk}(u), S_{kk}(v)\big] + \frac{(-1)^{[k]}}{u+v}\beta_k(u,v)\\ - \frac{1}{u^2-v^2}\big[S_{kk}(u), S_{kk}(v)\big] - \frac{1}{u^2-v^2}\beta_k(u,v),\end{aligned}$$

which becomes

$$\left(1 + \frac{1-2(-1)^{[k]}u}{u^2-v^2}\right)\big[S_{kk}(u), S_{kk}(v)\big] \equiv \frac{(-1)^{[k]}(u-v)-1}{u^2-v^2}\beta_k(u,v). \tag{5.2.11}$$

Hence, $[S_{kk}(u), S_{kk}(v)] \equiv 0$ if and only if $\beta_k(u,v) \equiv 0$. The base case $k = M+N$ is therefore satisfied since $\beta_{M+N}(u,v)$ is the empty sum, so we may suppose the induction hypothesis holds down to $k+1$.

Given $1 \le k < a \le M+N$, relations (5.1.31) assert the equivalences

$$\begin{aligned}\big[S_{aa}(u), S_{kk}(v)\big] \equiv \frac{1}{u-v}(-1)^{[a]}\big(S_{ka}(u)S_{ak}(v) - S_{ka}(v)S_{ak}(u)\big)\\ - \frac{1}{u^2-v^2}\big(\big[S_{kk}(u), S_{kk}(v)\big] + \beta_k(u,v)\big)\end{aligned}$$

and

$$\big[S_{kk}(v), S_{aa}(u)\big] \equiv \frac{1}{u^2-v^2}\big(\big[S_{aa}(u), S_{aa}(v)\big] + \beta_a(u,v)\big).$$

Therefore, since $[S_{aa}(u), S_{kk}(v)] = -[S_{kk}(v), S_{aa}(u)]$ we deduce the relation

$$\begin{aligned}&(-1)^{[a]}\big(S_{ka}(u)S_{ak}(v) - S_{ka}(v)S_{ak}(u)\big)\\ &\qquad\equiv \frac{1}{u+v}\big(\big[S_{kk}(u), S_{kk}(v)\big] - \big[S_{aa}(u), S_{aa}(v)\big] + \beta_k(u,v) - \beta_a(u,v)\big).\end{aligned}$$

Hence, by (5.2.10),

$$\begin{aligned}\beta_k(u,v) \equiv \frac{(-1)^{[k+1]}(M-k)-N}{u+v}\big(\big[S_{kk}(u), S_{kk}(v)\big] + \beta_k(u,v)\big)\\ - \frac{1}{u+v}\sum\nolimits_{a=k+1}^{M+N}(-1)^{[a]}\big(\big[S_{aa}(u), S_{aa}(v)\big] + \beta_a(u,v)\big),\end{aligned}$$

so we have

$$(u+v-(-1)^{[k+1]}(M-k)+N)\beta_k(u,v) \equiv ((-1)^{[k+1]}(M-k)-N)\big[S_{kk}(u),S_{kk}(v)\big]$$

since $[S_{aa}(u),S_{aa}(v)] \equiv 0 \equiv \rho_a(u,v)$ for $a>k$ by induction hypothesis. By combining the above equivalence with (5.2.11), one concludes $[S_{kk}(u),S_{kk}(v)] \equiv 0 \equiv \rho_k(u,v)$.

To finish the proof, it suffices to show $[S_{kk}(u),S_{ll}(v)] \equiv 0$ for $1 \le k < l \le M+N$. To this end, we realize by the relations (5.1.31) that

$$\big[S_{kk}(u),S_{ll}(v)\big] \equiv -\frac{1}{u^2-v^2}\big(\big[S_{ll}(u),S_{ll}(v)\big]+\beta_l(u,v)\big),$$

which is equivalent to zero by before. □

Leveraging the lemma just proven, we are now in position to prove the main theorem of this section.

**Theorem 5.2.3.** *Every finite-dimensional irreducible representation $V$ of the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ is a highest weight representation. The highest weight vector of $V$ is unique up to scalar multiple.*

*Proof.* Let $V$ denote a finite-dimensional irreducible representation of $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ and define the subspace

$$V^0 := \big\{v \in V \mid S_{ij}(u)v = 0 \text{ for all } 1 \le i < j \le M+N\big\} \tag{5.2.12}$$

We first establish that $V^0$ is non-trivial. Since the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_{M|N}$ lies within the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N},\mathcal{G})^{tw}$ under the embedding (5.1.18), one can consider the set of weight via the action of $\mathfrak{h}$ on $V$. There is a partial ordering '$\preccurlyeq$' on such set of weights via the rule that for any weights $\alpha,\beta \in \mathfrak{h}^*$, one has $\alpha \preccurlyeq \beta$ if and only if $\beta-\alpha$ is an $\mathbb{N}$-linear combination of positive roots in $\Phi+$.

Since the set $\{\mathcal{H}_k\}_{k=1}^{M+N}$ consists of pairwise commuting elements, their actions on $V$ form a family of pairwise commuting operators, implying that these operators must share a simultaneous eigenvector as $\dim V < \infty$. Such set of weights is finite, so $V$ must have a maximal weight $\mu$ with respect to the partial ordering '$\preccurlyeq$'.

Letting $v$ be a weight vector corresponding to $\mu$, the assertion follows if $v \in V^0$, so we may assume $v \notin V^0$ and therefore $S_{ij}^{(n)} v \neq 0$ for some $(i,j) \in \Lambda_{\Theta}^{+}$ and $n \in \mathbb{Z}^+$. However, since

$$\mathcal{H}_k S_{ij}^{(n)} v = T_{ij}^{(n)} \mathcal{H}_k v + \big[\mathcal{H}_k, S_{ij}^{(n)}\big] v,$$

we conclude from equation (5.2.2) that the weight of $S_{ij}^{(n)} v$ is of the form $\mu + \alpha$ for some positive root $\alpha \in \Phi^+$, contradicting the maximality of $\mu$ and proving the claim.

By Lemma 5.2.2, the actions of the generators $\{S_{kk}^{(n)} \mid 1 \leq k \leq M+N,\ n \in \mathbb{Z}^+\}$ form a family of pairwise commuting operators on $V^0$. As $V^0$ is a non-trivial subspace of $V$, there must exist a simultaneous eigenvector $0 \neq \xi \in V^0$ for such operators: $S_{kk}^{(n)} \xi = \lambda_k^{(n)} \xi$ for complex eigenvalues $\lambda_k^{(n)}$, $1 \leq k \leq M+N$, $n \in \mathbb{Z}^+$. Via the irreducibility of $V$, we conclude $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw} \xi = V$, and by collecting these eigenvalues into power series $\lambda_k(u) = \mathcal{G}_{kk} + \sum_{n=1}^{\infty} \lambda_k^{(n)} u^{-n}$ we observe the vector $\xi$ satisfies the conditions (5.2.4), so $V$ is a highest weight representation with highest weight vector $\xi$ and highest weight $(\lambda_k(u))_{k=1}^{M+N}$.

It remains to show that $\xi$ is unique up to scalar multiples. Recalling the PBW Theorem (5.1.10) for $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$, we fix a total order '$\preceq$' on the index set $\mathcal{K}$ (5.1.13) such that for any tuples $(i_1, j_1, n_1), (i_2, j_2, n_2), (i_3, j_3, n_3) \in \mathcal{K}$ satisfying $i_1 > j_1$, $i_2 = j_2$, $i_3 < j_3$, then $(i_1, j_1, n_1) \preceq (i_2, j_2, n_2) \preceq (i_3, j_3, n_3)$. Via this total ordering ordering, we conclude that $V$ is spanned by ordered elements of the form

$$S_{i_1 j_1}^{(n_1)} \cdots S_{i_k j_k}^{(n_k)} \xi, \tag{5.2.13}$$

where $k \in \mathbb{N}$, $i_a > j_a$, and $(i_a, j_a, n_a) \in \mathcal{K}$ for $1 \leq a \leq k$. By (5.2.2), the elements (5.2.13) are weight vectors with corresponding weights of the form

$$\mu + \sum_{a=1}^{k} (\varepsilon_{i_a} - \varepsilon_{j_a})$$

where $\mu$ is the linear functional on $\mathfrak{h}$ given by $\mathcal{H}_k \mapsto \frac{1}{2}(-1)^{[k]} \mathcal{G}_{kk} \lambda_k^{(1)}$. Hence, there is a weight space decomposition $V = \bigoplus_{\nu \in \mathfrak{h}^*} V_\nu$ where each weight $\nu \neq \mu$ is of the form $\mu - \sum_{a=1}^{k} (\varepsilon_{i_a} - \varepsilon_{j_a})$ for $i_a < j_a$, $a = 1, \ldots, k$. As a result, the space $V_\mu$ is 1-dimensional and is given by $V_\mu = \mathrm{span}_{\mathbb{C}}\{\xi\}$.

If $\tilde{\xi}$ is another highest weight vector of $V$ of highest weight $(\lambda_k(u))_{k=1}^{M+N}$, the weight space decomposition ensures that its weight must be equal to $\mu$. Hence, $\tilde{\xi} = c\xi$ for some $c \in \mathbb{C}^*$. $\square$

**Definition 5.2.4.** Given a tuple $\lambda(u) = (\lambda_k(u))_{k=1}^{M+N}$ of series of the form (5.2.5), the *Verma module* $M(\lambda(u))$ is the quotient:

$$M(\lambda(u)) := \mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw} / \mathcal{I}_{\lambda(u)}, \tag{5.2.14}$$

where $\mathcal{I}_{\lambda(u)}$ is the left graded ideal generated by the coefficients of the series $S_{ij}(u)$ where $1 \leq i < j \leq M+N$, and $S_{kk}(u) - \lambda_k(u)\mathbf{1}$ where $1 \leq k \leq M+N$.

When $M(\lambda(u))$ is non-trivial, it is a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ with highest weight $\lambda(u)$ and highest weight vector $\mathbf{1}_{\lambda(u)}$, the image of $\mathbf{1}$ in the canonical projection $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw} \to M(\lambda(u))$. Furthermore, if $L$ is a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ with highest weight $\lambda(u)$ and highest weight vector $\xi$, then, provided $M(\lambda(u))$ is non-trivial, there is a surjective $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$-module morphism $\varphi \colon M(\lambda(u)) \to L$ induced by the assignment $\mathbf{1}_{\lambda(u)} \mapsto \xi$; thus, $L \cong M(\lambda(u))/\ker\varphi$.

By (5.2.3),$\bigoplus_{\mu\in\mathfrak{h}^*} M(\lambda(u))_\mu$ is invariant under the action of $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$. Therefore, since $\mathbf{1}_{\lambda(u)}$ is contained in $M(\lambda(u))_{\lambda^{(1)}} \subset \bigoplus_{\mu\in\mathfrak{h}^*} M_\Theta(\lambda(u))_\mu$, where $\lambda^{(1)} \in \mathfrak{h}^*$ is the linear functional given by $\lambda^{(1)}(\mathcal{H}_k) = \lambda_k^{(1)}$, we have the weight space decomposition

$$M(\lambda(u)) = \bigoplus_{\mu\in\mathfrak{h}^*} M(\lambda(u))_\mu \tag{5.2.15}$$

and each weight $\mu$ is of the form $\lambda^{(1)} - \omega$, where $\omega$ is a $\mathbb{Z}^+$-linear combination of positive roots in $\Phi^+$. Indeed, recalling the PBW Theorem (5.1.10), we fix a total order $\preceq$ on the index set $\mathcal{K}$ (5.1.13) such that for any tuples $(i_1, j_1, n_1), (i_2, j_2, n_2), (i_3, j_3, n_3) \in \mathcal{K}$ satisfying $i_1 > j_1$, $i_2 = j_2$, $i_3 < j_3$, then $(i_1, j_1, n_1) \preceq (i_2, j_2, n_2) \preceq (i_3, j_3, n_3)$. Via this total ordering ordering, we conclude that $M(\lambda(u))$ is spanned by ordered elements of the form

$$S_{i_1 j_1}^{(n_1)} \cdots S_{i_k j_k}^{(n_k)} \mathbf{1}_{\lambda(u)}, \tag{5.2.16}$$

where $k \in \mathbb{N}$, $i_a > j_a$, and $(i_a, j_a, n_a) \in \mathcal{K}$ for $1 \leq a \leq k$. In particular, we conclude that $M(\lambda(u))_{\lambda^{(1)}}$ is 1-dimensional; $M(\lambda(u))_{\lambda^{(1)}} = \mathrm{span}_{\mathbb{C}}\{\mathbf{1}_{\lambda(u)}\}$.

Any submodule $P$ of $M(\lambda(u))$ also has a weight space decomposition $P = \bigoplus_{\mu\in\mathfrak{h}^*} P_\mu$, where $P_\mu = P \cap M(\lambda(u))_\mu$. Since $\dim M(\lambda(u))_{\lambda^{(1)}} = 1$, it necessarily follows that $P \subseteq \bigoplus_{\lambda^{(1)} \neq \mu \in \mathfrak{h}^*} M(\lambda(u))_\mu$ and so the sum of all proper submodules $K = \sum_{P < M(\lambda(u))} P$ is the unique maximal submodule of $M(\lambda(u))$.

We end this chapter by providing the following definition of an irreducible highest weight representation, which will be the types of representations used in the overall classification of the set $\mathrm{Rep}^{\mathrm{irr}}_{\mathrm{fd}}(\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw})/_\sim$:

**Definition 5.2.5.** When the Verma module $M(\lambda(u))$ is non-trivial, we define the *irreducible highest weight representation* $L(\lambda(u))$ of $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ with highest weight $\lambda(u)$ as the quotient of $M(\lambda(u))$ by its unique maximal proper submodule.

The next goal in the story of the representation theory for the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ is to classify the necessary and sufficient conditions for the non-triviality of the Verma module $M(\lambda(u))$. Addressing such question is unfortunately out of the scope of the dissertation, which is the cause of future research in this area.

# Chapter 6

# Conclusion

This dissertation addressed three different topics in super Yangian theory. In summary, Chapters 2 and 3 investigated Yangians of orthosymplectic Lie superalgebras, while Chapter 4 considered Yangians of periplectic Lie superalgebras and Chapter 5 introduced twisted super Yangians of type AIII.

Chapter 2 provided a detailed exposition on the algebraic structure of the orthosymplectic Yangians $\mathrm{Y}(\mathfrak{osp}_{M|N})$, which involved proving a Poincaré-Birkhoff-Witt-type theorem in detail that has not yet occurred in the literature. In Chapter 3, we established many necessary conditions for the irreducible representations of $\mathrm{X}(\mathfrak{osp}_{M|N})$ to be finite-dimensional by formulating a suitable highest weight theory.

Later in Chapter 4, we recalled the strange Yangians $\mathrm{Y}(\mathfrak{s}_N)$ for $\mathfrak{s}_N = \mathfrak{p}_N, \mathfrak{q}_N$ as originally defined by M. Nazarov in [Naz92]. We proved a Poincaré-Birkhoff-Witt-type theorem for the Yangian $\mathrm{Y}(\mathfrak{p}_N)$ by adapting the arguments used to show a similar result for its counterpart $\mathrm{Y}(\mathfrak{q}_N)$ in the paper [Naz99]. Lastly, we defined the twisted Yangians $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ of type AIII in Chapter 5 and established that they can also be realized as reflection superalgebras subject to an additional unitary constraint. Moreover, we founded a highest weight theory for $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$ and proven that every finite-dimensional irreducible representation must be highest weight.

To conclude this chapter and dissertation, we survey some natural questions that arise from our work which instigates future research directions.

### 6.0.1 The classification of finite-dimensional irreducible representations and the universal $R$-matrix of $\mathrm{Y}(\mathfrak{osp}_{M|N})$

The main research priority on these Yangians is to obtain sufficient conditions for the irreducible representations of $\mathrm{X}(\mathfrak{osp}_{M|N})$ to be finite-dimensional. This will be achieved if one is able to finalize the construction of the remaining type II fundamental representations as carried out in §3.2.4. To prove such sufficient conditions are also necessary, there will need to be further restrictions on the roots of the Drinfel'd polynomials $\widetilde{Q}(u)$ and $Q(u)$ associated to any finite-dimensional irreducible representation. If both of these tasks can be completed, a full classification of the sets $\mathrm{Rep}_{\mathrm{fd}}^{\mathrm{irr}}(\mathrm{X}(\mathfrak{osp}_{M|N}))_{/\sim}$ and $\mathrm{Rep}_{\mathrm{fd}}^{\mathrm{irr}}(\mathrm{Y}(\mathfrak{osp}_{M|N}))_{/\sim}$ should be possible; we refer the reader to the Conjecture 3.2.22 and Conjecture 3.2.23.

When $\mathfrak{g}$ is a simple Lie algebra, it has been known since [Dri85] that a universal $R$-matrix exists for the Yangian $\mathrm{Y}(\mathfrak{g})$:

$$\mathcal{R}(u) = \mathbf{1} + \sum_{k=1}^{\infty} \mathcal{R}_k u^{-k} \in (\mathrm{Y}(\mathfrak{g}) \otimes \mathrm{Y}(\mathfrak{g}))[\![u^{-1}]\!].$$

In fact, a recent constructive proof of the universal $R$-matrix in this case has appeared in the article [GLW21]. The mentioned proof, however, utilizes Drinfel'd's current presentation of the Yangian as opposed to the *RTT* formalism used predominantly in this dissertation.

As initially discussed in Chapter 1, the question of whether or not such a universal $R$-matrix exists for the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ currently remains open. Recent progress has been achieved in formulating an analogue of Drinfel'd's current presentation for the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ in the works [Mol23a, MR23]; hence, a natural research direction would be to adapt the arguments used in the constructive proof of the $R$-matrix to the case of the orthosymplectic Yangian $\mathrm{Y}(\mathfrak{osp}_{M|N})$ utilizing Drinfel'd's current presentation.

Another research direction is the construction of the *Yangian double*: the adaptation of the Drinfel'd double in Hopf algebra theory to such infinite-dimensional quantum groups. In the paper [JYL20], a construction of the Yangian doubles $\mathrm{DY}(\mathfrak{so}_M)$ and $\mathrm{DY}(\mathfrak{sp}_N)$ were provided via the *RTT* formalism. A direct adaptation of such constructions in the aforementioned paper to the supersymmetric setting should be blueprint for defining the Yangian double $\mathrm{DY}(\mathfrak{osp}_{M|N})$.

### 6.0.2 Representation theory for Yangians of type $P$

In Chapter 4, we investigated much of the algebraic structure of the Yangian $\mathrm{Y}(\mathfrak{p}_N)$; however, we did not examine any of its representation theory. In [Naz99, §5], Nazarov constructed functors between the representation categories of $\mathrm{Y}(\mathfrak{q}_N)$ and the *degenerate affine Sergeev algebras*, thereby constructing a wide array of irreducible representations for this Yangian. Accordingly, one could attempt to imitate the construction of such a representation functor between the representation categories of $\mathrm{Y}(\mathfrak{p}_N)$ and some other superalgebra $\mathcal{A}$.

In fact, the author has already performed calculations on this question when $\mathcal{A}$ is the *degenerate affine periplectic Brauer algebra* $\mathfrak{B}_d^{\mathrm{aff}}$, which is denoted $\widehat{P}_{\bar{d}}$ in the paper [CP18]. However, certain difficulties arise in this case that forces one to take a certain unnatural quotient of the algebra $\mathfrak{B}_d^{\mathrm{aff}}$ which unfortunately diminishes the value of the result. Furthermore, the relations between the super permutation operator $P$ and the matrix $Q^{\mathsf{P}}$ are somewhat more complicated than the comparative relations between $P$ and $Q^{\mathsf{Q}}$, which creates complications when attempting to construct a faithful adaptation of the functor presented by Nazarov.

Of course, a direct investigation into the classification of finite-dimensional irreducible representations of $\mathrm{Y}(\mathfrak{p}_N)$ is a possible future research venture. Similar to the orthosymplectic Yangians, one would hope to first establish a tensor product decomposition

$$\mathrm{X}(\mathfrak{p}_N) \cong \mathrm{ZX}(\mathfrak{p}_N) \otimes \mathrm{Y}(\mathfrak{p}_N),$$

where $\mathrm{ZX}(\mathfrak{p}_N)$ is the supercenter of the extended Yangian $\mathrm{X}(\mathfrak{p}_N)$, thereby allowing one to focus on the representation theory of $\mathrm{X}(\mathfrak{p}_N)$. However, such a decomposition is yet to be proven.

A construction of the Yangian double $\mathrm{DY}(\mathfrak{p}_N)$ may also be possible in a similar way to the Yangian double in [JYL20] via the *RTT* formalism. In [Naz99, §4], Nazarov does define the Yangian double $\mathrm{DY}(\mathfrak{q}_N)$ as the superalgebra generated by $\mathrm{Y}(\mathfrak{q}_N)$ and its *dual* $\mathrm{Y}^*(\mathfrak{q}_N)$. However, adapting many of the results involving $\mathrm{Y}(\mathfrak{q}_N)$ and its dual to the periplectic case can prove to be difficult due to the aforementioned relationship between the matrices $P$ and $Q^{\mathsf{P}}$.

### 6.0.3 The supercenter and representation theory for twisted super Yangians of type AIII

The last research direction is on the topic of the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$. The notable omission on the algebraic structure of these twisted super Yangians in Chapter 5 is the description of its supercenter. When $N = 0$, it is known by [MR02, § 3] that the center of $\mathrm{Y}(\mathfrak{gl}_{M|0}, \mathcal{G})^{tw} \cong \mathrm{Y}(\mathfrak{gl}_M, \mathcal{G})^{tw}$ is determined by the *Sklyanin determinant* $\operatorname{sdet} S(u)$ which itself is given in terms of the *quantum determinant* $\operatorname{qdet} T(u)$, where $T(u)$ is the generating matrix of $\mathrm{Y}(\mathfrak{gl}_M)$.

It is known that the quantum determinant generates the center of $\mathrm{Y}(\mathfrak{gl}_M)$, whereas its super-analogue, the *quantum Berezinian* (see [Gow07, §7]), generates the supercenter of $\mathrm{Y}(\mathfrak{gl}_{M|N})$. To describe the supercenter of the twisted Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N}, \mathcal{G})^{tw}$, one would therefore aim to construct a super-analogue of the Sklyanin determinant in terms of the quantum Berezinian in lieu of the quantum determinant.

After obtaining the description of the supercenter of these twisted super Yangians, the next goal is to continue the investigation into their finite-dimensional irreducible representations. In particular, the next immediate result to attain is the necessary and sufficient conditions for the non-triviality of their Verma modules $M(\lambda(u))$. Certain necessary conditions are already determined by the author, but the proof for sufficiency will likely require the use of the quantum Berezinian in accordance with the methodology used in the proof of the non-super case [MR02, Theorem 4.2].

Ultimately, by following the exposition of [MR02], one would have to address the classification of the finite-dimensional irreducible representations for the low rank cases $\mathrm{Y}(\mathfrak{gl}_{1|1}, \mathcal{G})^{tw}$ and $\mathrm{Y}(\mathfrak{gl}_{2|1}, \mathcal{G})^{tw}$, but this will require further investigation.

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